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2010 J. Phys. A: Math. Theor. 43 065202

(http://iopscience.iop.org/1751-8121/43/6/065202)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 065202 (40pp) doi:10.1088/1751-8113/43/6/065202

The conservation of the Hamiltonian structures in the deformations of the Whitham systems

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Received 10 November 2009, in final form 18 December 2009
Published 20 January 2010
Online at stacks.iop.org/JPhysA/43/065202

Abstract
We consider the construction of the deformed Whitham system for the KdV equation in the one-phase case and investigate the conservation of the Hamiltonian properties in this situation. It is shown then that both the Gardner–Zakharov–Faddeev and the Magri brackets give the deformed Dubrovin–Novikov brackets (the brackets of Dubrovin–Zhang type) for the deformed Whitham system constructed by our procedure. The general approach used in the paper gives a scheme for the averaging of the Poisson structures in the general situation.

PACS numbers: 02.30.Ik, 02.30.Jr

1. Introduction

We consider the conservation of local field-theoretical Hamiltonian structures in the method of deformations of the Whitham systems. As is well known, the Whitham method is connected with the slow modulations of parameters of (one- or multi-phase) periodic or quasi-periodic solutions of PDEs, while the Whitham system itself rules the behavior of the modulated parameters such as the functions of time and spatial variables. The Whitham system is usually written as a system of hydrodynamic type:

$$U^{\nu}_{T} = V^{\nu}_{\mu}(\mathbf{U}) U^{\mu}_{X} \tag{1.1}$$

and gives the main term in the connection of the time and spatial derivatives of parameters $U^{\nu}(X, T)$. The variables T and X usually represent the 'slow' time and spatial variables $T = \epsilon t$, $X = \epsilon x$ connected with the variables t and x through the small parameter ϵ. The Whitham system (1.1) is then a homogeneous system of the hydrodynamic type connecting the first derivatives of the slow modulated parameters. Many studies on the different aspects and numerous applications of the Whitham method have been carried out and the Whitham method is now considered as one of the classical methods used to investigate nonlinear systems.

Different properties of the Whitham equations were investigated by many authors (see for instance [1–3, 6–13, 24, 29–34, 36–46, 50–52, 54–56, 58–63, 65–72]). Thus, it was pointed out by Whitham [69–71] that the Whitham system (1.1) has a local Lagrangian structure when the initial system has a local Lagrangian structure

$$\delta \iint \mathcal{L}(\varphi, \varphi_t, \varphi_x, \ldots)\, \mathrm{d}x\, \mathrm{d}t = 0$$

on the initial phase space $\{\varphi(x,t)\}$.

The procedure for constructing the Lagrangian formalism for the Whitham system (1.1) is given by the averaging of the Lagrangian function $\mathcal{L}$ on the family of m-phase solutions of the initial system. Let us also note that in the case of the presence of additional parameters n^l, the additional method of the Whitham pseudo-phases should be used.

The important procedure of the averaging of local field-theoretical Hamiltonian structures was suggested by Dubrovin and Novikov [11–13, 58]. The Dubrovin–Novikov procedure gives the local field-theoretical Hamiltonian formalism for the Whitham system (1.1) when the initial system has a local Hamiltonian formalism of general type. The Dubrovin–Novikov bracket for the Whitham system has a general form

$$\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(\mathbf{U})\, \delta'(X-Y) + b^{\nu\mu}_\lambda(\mathbf{U}) U^\lambda_X \delta(X-Y) \qquad (1.2)$$

and was called the local Poisson bracket of hydrodynamic type. The theory of the brackets (1.2) is closely related with differential geometry [11–13] and is connected with different coordinate systems in the (pseudo) Euclidean spaces. Let us also mention that in the last few years the important weakly nonlocal generalizations of Dubrovin–Novikov brackets (Mokhov–Ferapontov bracket and Ferapontov brackets) were introduced and studied [25–28, 53, 57, 59, 60].

Over the past few years the theory of deformations of systems (1.1) and the Poisson brackets (1.2) was intensively studied [14–23, 47–49]. The ϵ-deformations of systems of the hydrodynamic type (1.1) and of brackets (1.2) give the 'dispersive' corrections to (1.1) and (1.2) and are represented usually as the formal series in the powers of ϵ with the higher derivatives of the parameters $\mathbf{U}$. Let us mention that the theory of the compatible Poisson brackets (1.2) and their deformations demonstrate very nontrivial structures and is now considered as one of the general approaches in the classification of integrable hierarchies.

We will consider here the deformations of systems (1.1) and the Poisson brackets (1.2) connected immediately with the Whitham method for the slow-modulated parameters. As far as we know, the idea of considering the dispersive Whitham systems first appeared in the paper by Ablowitz and Benney [1], where the dispersive character of the higher corrections in the Whitham approach was pointed out. The regular procedure of deformation of the Whitham systems was constructed in [55] in connection with the theory of deformations of systems of hydrodynamic type developed in [19, 20]. In [56] a special modification of the deformation procedure which gives a regular transition from the linear to nonlinear systems was also suggested. We will need in this paper the modification of the deformation procedure considered in [56] since solutions with the vanishing amplitude of oscillations will arise in our consideration.

The main goal of this paper is to prove the conservation of local field-theoretical Hamiltonian structures in the method of deformations of the Whitham systems which is considered in the example of the one-phase modulated solutions of the KdV equation. Namely, we suggest here a scheme of the 'averaging' of local field-theoretical Poisson brackets giving the deformed Dubrovin–Novikov brackets for the deformed Whitham systems (1.1). The procedure considered here is based on the Dirac procedure of restricting the Poisson bracket on a sub-manifold which provides the Jacobi identity for the 'averaged' Poisson structures.

In sections 2 and 3, we describe the scheme of deformation of the Whitham system for the KdV equation giving the dispersive corrections to the standard system of Whitham in this situation. In section 4, we consider immediately the averaging of two local Hamiltonian structures for KdV and prove the existence of two deformed brackets (1.2) for the deformed Whitham system. Finally, in section 5, a scheme of the averaging of the local Lagrangian structures in the method of deformations of the Whitham system is also considered. The construction used here has in fact a general character and can be used in analogous form for different systems of PDEs.

2. The Whitham method and the deformation scheme

As is well known, the Whitham method [69–71] is connected with the slow modulations of periodic or quasiperiodic m-phase solutions of nonlinear systems:

$$F^i(\varphi, \varphi_t, \varphi_x, \ldots) = 0, \qquad i = 1, \ldots, n, \qquad \varphi = (\varphi^1, \ldots, \varphi^n), \tag{2.1}$$

which are represented usually in the form

$$\varphi^i(x, t) = \Phi^i(\mathbf{k}(\mathbf{U})x + \boldsymbol{\omega}(\mathbf{U})t + \boldsymbol{\theta}_0, \mathbf{U}). \tag{2.2}$$

In these notations, the functions $\mathbf{k}(\mathbf{U})$ and $\boldsymbol{\omega}(\mathbf{U})$ play the role of the 'wave numbers' and 'frequencies' of m-phase solutions and $\boldsymbol{\theta}_0$ are the initial phase shifts. The parameters of the solutions $\mathbf{U} = (U^1, \ldots, U^N)$ can be chosen in an arbitrary way; however, we assume that they do not change under arbitrary shifts of the initial phases $\boldsymbol{\theta}_0$ of solutions.

The functions $\Phi^i(\boldsymbol{\theta})$ satisfy the system

$$F^i(\Phi, \omega^\alpha \Phi_{\theta^\alpha}, k^\beta \Phi_{\theta^\beta}, \ldots) \equiv 0, \qquad i = 1, \ldots, n \tag{2.3}$$

and we choose for every $\mathbf{U}$ some function $\Phi(\boldsymbol{\theta}, \mathbf{U})$ as having 'zero initial phase shifts'. The full set of m-phase solutions of (2.1) can then be represented in the form (2.2). For m-phase solutions of (2.1) we have then $\mathbf{k}(\mathbf{U}) = (k^1(\mathbf{U}), \ldots, k^m(\mathbf{U}))$, $\boldsymbol{\omega}(\mathbf{U}) = (\omega^1(\mathbf{U}), \ldots, \omega^m(\mathbf{U}))$, $\boldsymbol{\theta}_0 = (\theta^1, \ldots, \theta^m)$, where $\mathbf{U} = (U^1, \ldots, U^N)$ are parameters of the solution. We also require that all the functions $\Phi^i(\boldsymbol{\theta}, \mathbf{U})$ are 2π-periodic with respect to every θ^α, $\alpha = 1, \ldots, m$.

In the Whitham approach, the parameters $\mathbf{U}$ become slow functions of x and t: $\mathbf{U} = \mathbf{U}(X, T)$, where $X = \epsilon x$, $T = \epsilon t$ ($\epsilon \to 0$).

The functions $\mathbf{U}(X, T)$ should satisfy in this case some system of differential equations (Whitham system) which makes the construction of the corresponding asymptotic solution possible. More precisely (see [50]), we try to find the asymptotic solutions

$$\varphi^i(\boldsymbol{\theta}, X, T) = \sum_{k \geqslant 0} \Psi^i_{(k)}\left(\frac{\mathbf{S}(X, T)}{\epsilon} + \boldsymbol{\theta}, X, T\right)\epsilon^k \tag{2.4}$$

(where all $\boldsymbol{\Psi}_{(k)}$ are 2π-periodic in $\boldsymbol{\theta}$) which satisfy system (2.1), i.e.

$$F^i(\varphi, \epsilon\varphi_T, \epsilon\varphi_X, \ldots) = 0, \qquad i = 1, \ldots, n.$$

The function $\mathbf{S}(X, T) = (S^1(X, T), \ldots, S^m(X, T))$ is called a 'modulated phase' of solution (2.4).

It is easy to see that the function $\boldsymbol{\Psi}_{(0)}(\boldsymbol{\theta}, X, T)$ should belong to the family of m-phase solutions of (2.1) at every X and T. So we have

$$\boldsymbol{\Psi}_{(0)}(\boldsymbol{\theta}, X, T) = \boldsymbol{\Phi}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(X, T), \mathbf{U}(X, T)) \tag{2.5}$$

and

$$S^\alpha_T(X, T) = \omega^\alpha(\mathbf{U}), \qquad S^\alpha_X(X, T) = k^\alpha(\mathbf{U})$$

as follows from the substitution of (2.4) into system (2.1).

The functions $\mathbf{\Psi}_{(k)}(\boldsymbol{\theta}, X, T)$ are defined from the linear systems

$$\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(X, T)\Psi^j_{(k)}(\boldsymbol{\theta}, X, T) = f^i_{(k)}(\boldsymbol{\theta}, X, T), \tag{2.6}$$

where $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(X, T)$ is a linear operator given by the linearization of system (2.3) on solution (2.5). The resolvability conditions of systems (2.6) can be written as the orthogonality conditions of the functions $\mathbf{f}_{(k)}(\boldsymbol{\theta}, X, T)$ to all the 'left eigenvectors' (the eigenvectors of the adjoint operator) of the operator $\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(X, T)$ corresponding to zero eigenvalues. The resolvability conditions of (2.6) for $k = 1$

$$\hat{L}^i_{j[\mathbf{U},\boldsymbol{\theta}_0]}(X, T)\Psi^j_{(1)}(\boldsymbol{\theta}, X, T) = f^i_{(1)}(\boldsymbol{\theta}, X, T) \tag{2.7}$$

together with the relations $k^\alpha_T = \omega^\alpha_X$ give the Whitham system for m-phase solutions of (2.1) which plays the central role in the slow modulations approach.

Let us say that the resolvability conditions of (2.6) can in fact be quite complicated in a general multi-phase case. Indeed, we need to investigate the eigenspaces of the operators $\hat{L}_{[\mathbf{U},\boldsymbol{\theta}_0]}$ and $\hat{L}^\dagger_{[\mathbf{U},\boldsymbol{\theta}_0]}$ on the space of 2π-periodic functions which can be non-trivial in the multi-phase situation. Thus, even the dimensions of kernels of $\hat{L}_{[\mathbf{U},\boldsymbol{\theta}_0]}$ and $\hat{L}^\dagger_{[\mathbf{U},\boldsymbol{\theta}_0]}$ can depend, in a non-smooth way, on the values of $\mathbf{U}$ so we can have a rather complicated picture on the $\mathbf{U}$-space [6–8].

These difficulties do not usually appear in the one-phase situation ($m = 1$) where the behavior of eigenvalues of $\hat{L}_{[\mathbf{U},\boldsymbol{\theta}_0]}$ and $\hat{L}^\dagger_{[\mathbf{U},\boldsymbol{\theta}_0]}$ is usually regular.

In this section we are going to consider a scheme of deformation of the Whitham system giving 'dispersive' corrections to the system of hydrodynamic type which describe the higher corrections to the corresponding asymptotic solutions. We are going to use here the one-phase modulated solutions of the KdV equation as a basic example throughout the paper, so let us now consider the KdV equation

$$\varphi_t + \varphi\varphi_x + \varphi_{xxx} = 0. \tag{2.8}$$

It has a family of exact solutions of the form

$$\varphi(x, t) = \Phi(k(\mathbf{U})x + \omega(\mathbf{U})t, \mathbf{U}), \tag{2.9}$$

where the functions $\Phi(\theta, \mathbf{U})$ depending on three real parameters U^1, U^2, U^3 are 2π-periodic in θ.

As we pointed out already the Whitham modulation theory gives a prescription for finding approximate solutions to KdV in the form

$$\varphi \simeq \Phi\left(\frac{S(X, T)}{\epsilon}; U^1(X, T), U^2(X, T), U^3(X, T)\right), \tag{2.10}$$

where ϵ is a small parameter, $X = \epsilon x$, $T = \epsilon t$ are slow variables, and the dependence of the parameters $U^\nu = U^\nu(X, T)$ is determined from certain system of the first order quasilinear equations of the form

$$U^\nu_T = V^\nu_\mu(U^1, U^2, U^3)U^\mu_X, \qquad \nu, \mu = 1, \ldots, 3. \tag{2.11}$$

The phase function $S(X, T)$ is determined by quadratures

$$S_X(X, T) = k(X, T), \qquad S_T(X, T) = \omega(X, T).$$

The deformed Whitham equations will arise in the description of solutions to (2.8) in the form

$$\varphi = \Phi\left(S(X, T) + \theta; \mathbf{U}(X, T)\right) + \sum_{l\geqslant 1} \Phi_{(l)}(S(X, T) + \theta; X, T), \tag{2.12}$$

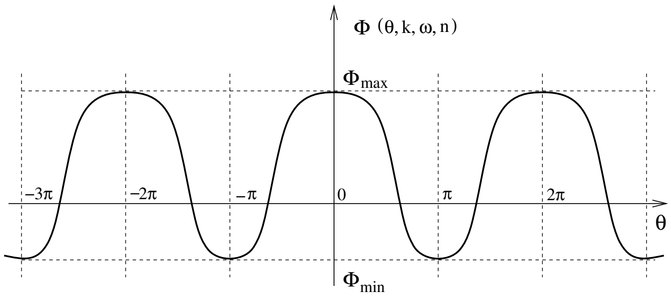

Figure 1. The function $\Phi(\theta, k, \omega)$ having zero initial phase shift.

where the functions

$$\Phi_{(l)}(\theta; X, T) = \Phi_l(\theta; \mathbf{U}, \mathbf{U}_X, \mathbf{U}_{XX}, \ldots, \mathbf{U}^{(l)})$$

2π-periodic in θ are graded homogeneous differential polynomials in $\mathbf{U}_X$, $\mathbf{U}_{XX}$, etc with coefficients being smooth functions of $\mathbf{U} = (U^1, U^2, U^3)$. The gradation is defined by the rule

$$\deg \partial_X^m \mathbf{U} = m, \qquad m = 1, 2, \ldots.$$

As usual the degree of the product of homogeneous differential polynomials is equal to the sum of their degrees. We use here the notations X and T just to emphasize that the functions $\mathbf{U}(X, T)$ are 'slow' functions of spatial and time variables. At the moment we do not write the small parameter ϵ specifically; it will be reintroduced later.

It will be convenient to choose a particular system of coordinates U^1, U^2, U^3 in the space of traveling wave solutions $\Phi(\theta; \mathbf{U})$. We denote them as

$$\mathbf{U} = (k, \omega, n),$$

where k and ω are the wave number and the frequency, and n is the mean value of Φ. The ODE for the function Φ

$$\omega\Phi_\theta + k\Phi\Phi_\theta + k^3\Phi_{\theta\theta\theta} = 0 \tag{2.13}$$

can be integrated by quadratures

$$\sqrt{\frac{k^3}{2}} \int_{a_3} \frac{\mathrm{d}\Phi}{\sqrt{-k\Phi^3/6 - \omega\Phi^2/2 + A\Phi + B}} = \theta,$$

where a_3 is the third zero of the cubic polynomial $-k\Phi^3/6 - \omega\Phi^2/2 + A\Phi + B$ according to the normalization shown in figure 1. The dependence on the parameters of the coefficients of the polynomials $A = A(k, \omega, n)$, $B = B(k, \omega, n)$ is determined from the equations

$$\sqrt{\frac{k^3}{2}} \oint \frac{\mathrm{d}\Phi}{\sqrt{-k\Phi^3/6 - \omega\Phi^2/2 + A\Phi + B}} = 2\pi$$

$$\sqrt{\frac{k^3}{2}} \oint \frac{\Phi\mathrm{d}\Phi}{\sqrt{-k\Phi^3/6 - \omega\Phi^2/2 + A\Phi + B}} = 2\pi n.$$

We also fix the initial phase shift of the functions $\Phi(\theta, k, \omega, n)$ in such a way that every $\Phi(\theta, k, \omega, n)$ has a local maximum at the point $\theta = 0$ (see figure 1).

It is well-known that the function $-\Phi(kx + \omega t, \mathbf{U})/6$ represents the one-gap potential for the Schrödinger operator

$$\hat{L} = -\frac{\mathrm{d}}{\mathrm{d}x^2} - \frac{\varphi}{6} \qquad (2.14)$$

while the KdV equation can be written in the Lax representation

$$\frac{\mathrm{d}\hat{L}}{\mathrm{d}t} = [\hat{A}, \hat{L}],$$

where

$$\hat{A} = -4\frac{\mathrm{d}^3}{\mathrm{d}x^3} - \frac{1}{2}\left(\varphi\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\varphi\right).$$

Let us say that the integrability of the KdV equation will be convenient in some aspects of our considerations. However, the general questions considered here are not connected with the integrability of the KdV equation and are applicable for the non-integrable examples as well.

It is also well known that the solution $\Phi(kx + \omega t, \mathbf{U})$ can be represented in the form

$$\Phi(kx + \omega t, \mathbf{U}) = \frac{2a}{s^2}\mathrm{d}n^2\left[\left(\frac{a}{6s^2}\right)^{1/2}(x - Vt), s\right] + \gamma$$

$$V = \frac{2a}{3s^2}(2 - s^2) + \gamma,$$

where s is the modulus of the Jacobi elliptic function $\mathrm{d}n(u, s)$, $0 \leqslant s \leqslant 1$. The value $2a$ plays the role of the amplitude of oscillations for the periodic solution and the values (k, ω, n) can be expressed in terms of the parameters (a, s, γ) in the following way:

$$k = \frac{\pi}{K(s)}\left(\frac{a}{6s^2}\right)^{1/2}, \qquad \omega = -Vk = -\frac{4\pi}{K(s)}(2 - s^2)\left(\frac{a}{6s^2}\right)^{3/2} - \frac{\gamma\pi}{K(s)}\left(\frac{a}{6s^2}\right)^{1/2},$$

$$n = \gamma + \frac{2aE(s)}{s^2K(s)},$$

where $K(s)$ and $E(s)$ are the elliptic integrals of the first and the second kind, respectively.

We can also write

$$\Phi(\theta, a, s, \gamma) = \frac{2a}{s^2}\,\mathrm{d}n^2\left(\frac{K(s)}{\pi}\theta, s\right) + \gamma$$

for our normalization of the functions $\Phi(\theta, \mathbf{U})$. Let us also note that the parameters (a, s, γ) are connected with the energy band edges (r_1, r_2, r_3) of the operator (2.14) by the formulas

$$r_2 - r_1 = a, \qquad \frac{r_2 - r_1}{r_3 - r_1} = s^2, \qquad r_1 + r_2 - r_3 = \gamma$$

$(r_3 > r_2 > r_1)$.

The total function

$$\Phi^{(\mathrm{tot})}(\theta, X, T) = \sum_{l\geqslant 0}\Phi_{(l)}(\theta, X, T) = \phi(\theta - S(X, T), X, T)$$

satisfies the equation

$$\begin{aligned} &S_T\Phi^{(\mathrm{tot})}_\theta + S_X\Phi^{(\mathrm{tot})}\Phi^{(\mathrm{tot})}_\theta + (S_X)^3\,\Phi^{(\mathrm{tot})}_{\theta\theta\theta} \\ &\quad + \Phi^{(\mathrm{tot})}_T + \Phi^{(\mathrm{tot})}\Phi^{(\mathrm{tot})}_X + 3S_X^2\Phi^{(\mathrm{tot})}_{\theta\theta X} + 3S_XS_{XX}\Phi^{(\mathrm{tot})}_{\theta\theta} \\ &\quad + 3S_X\Phi^{(\mathrm{tot})}_{\theta XX} + 3S_{XX}\Phi^{(\mathrm{tot})}_{\theta X} + S_{XXX}\Phi^{(\mathrm{tot})}_\theta + \Phi^{(\mathrm{tot})}_{XXX} = 0. \end{aligned} \qquad (2.15)$$

This yields linear equations for the functions $\Phi_{(l)}(\theta, X, T)$ for $l \geqslant 1$. In particular, the function $\Phi_{(1)}(\theta, X, T)$ satisfies the equation

$$\omega\Phi_{(1)\theta} + k\Phi_{(1)}\Phi_\theta + k\Phi_{(1)\theta}\Phi + k^3\Phi_{(1)\theta\theta\theta} = f_{(1)}(\theta, X, T), \tag{2.16}$$

where

$$f_{(1)}(\theta, X, T) = -\Phi_T^{[1]} - \Phi\Phi_X - 3k^2\Phi_{\theta\theta X} - 3kk_X\Phi_{\theta\theta} \tag{2.17}$$

and the notation $^{[1]}$ means that we consider just the part of Φ_T having degree 1 according to our definition.

Denote by $\hat{L}_{[X,T]}$ the linear operator:

$$\hat{L}_{[X,T]} = \omega\frac{\partial}{\partial\theta} + k\frac{\partial}{\partial\theta}\Phi + k^3\frac{\partial^3}{\partial\theta^3}. \tag{2.18}$$

We can rewrite (2.17) in the form

$$\hat{L}_{[X,T]}\Phi_{(1)} = f_{(1)}.$$

In the same way, we have the analogous systems for the functions $\Phi_{(l)}(\theta, X, T)$ having the form

$$\hat{L}_{[X,T]}\Phi_{(l)} = \omega\Phi_{(l)\theta} + k\Phi_{(l)}\Phi_\theta + k\Phi_{(l)\theta}\Phi + k^3\Phi_{(l)\theta\theta\theta} = f_{(l)}(\theta, X, T), \tag{2.19}$$

where $f_{(l)}(\theta, X, T)$ are the discrepancies having degree l.

The functions $k(X, T) = S_X$, $\omega(X, T) = S_T$ and $n(X, T)$ must satisfy the 'deformed Whitham system'

$$\begin{aligned} k_T &= \omega_X \\ \omega_T &= \sum_{l\geqslant 1}\sigma_{(l)}\,(k, \omega, n, k_X, \omega_X, n_X, \ldots) \\ n_T &= \sum_{l\geqslant 1}\eta_{(l)}\,(k, \omega, n, k_X, \omega_X, n_X, \ldots)\,, \end{aligned} \tag{2.20}$$

where all $\sigma_{(l)}$, $\eta_{(l)}$ are graded homogeneous differential polynomials in $(k, \omega, n, k_X, \omega_X, n_X, \ldots)$ of the degree l.

It is easy to see that relations (2.20) give in fact a possibility of representing in the form of homogeneous differential polynomials any expression like $k_{TX\ldots X}$, $\omega_{TX\ldots X}$, $n_{TX\ldots X}$, and even $k_{T\ldots TX\ldots X}$, $\omega_{T\ldots TX\ldots X}$, $n_{T\ldots TX\ldots X}$ iterating the subsequent substitution of the series (2.20). (The last property will not be necessary for the KdV equation.)

According to (2.20) all the time derivatives like Φ_T, $\Phi_{(l)T}$ can also be represented as the sum of homogeneous components

$$\Phi_{(l)T} = \Phi_{(l)T}^{[l]} + \Phi_{(l)T}^{[l+1]} + \Phi_{(l)T}^{[l+2]} + \cdots,$$

where the functions $\Phi_{(l)T}^{[s]}$ are differential polynomials of $(k, \omega, n, k_X, \omega_X, n_X, \ldots)$ of the degree s.

We impose the following orthogonality conditions on the discrepancies $f_{(l)}(\theta, X, T)$:

$$\int_0^{2\pi} f_{(l)}\frac{d\theta}{2\pi} = 0, \qquad \int_0^{2\pi}\Phi f_{(l)}\frac{d\theta}{2\pi} = 0 \tag{2.21}$$

and also the 'normalization' conditions

$$\int_0^{2\pi}\Phi_\theta\,\Phi_{(l)}\frac{d\theta}{2\pi} = 0, \qquad \int_0^{2\pi}\Phi_{(l)}\frac{d\theta}{2\pi} = 0 \tag{2.22}$$

for the functions $\Phi_{(l)}(\theta, X, T)$ defined from (2.19) modulo the linear combinations $a(X, T)\Phi_\theta + b(X, T)\Phi_n$.

For determination of $\sigma_{(l)}$, $\eta_{(l)}$ we use the system (2.20) to remove all time derivatives of (k, ω, n) after the substitution of (2.12) into (2.8) in order to represent (2.15) in the graded form.

The functions $\sigma_{(l)}$, $\eta_{(l)}$ arising in (2.20) are found from the compatibility conditions of systems (2.19) in the lth order. It can be shown that conditions (2.21) and (2.22) define uniquely all the expressions $\sigma_{(l)}$, $\eta_{(l)}$ and the corrections $\Phi_{(l)}$, $l \geqslant 1$.

So, our prescription for deriving system (2.20) is based on the following three conditions.

(I) All the functions $\Phi(\theta; k, \omega, n)$ are chosen in the way shown in figure 1.

(II) The modulated phase $S(X, T)$ is connected with the parameters (k, ω, n) by the relations

$$S_T(X,T) = \omega(X,T), \qquad S_X(X,T) = k(X,T).$$

(III) All the higher corrections $\Phi_{(l)}(\theta, X, T)$, $l \geqslant 1$, satisfy the normalization conditions (2.22).

According to the statements above, system (2.20) is uniquely defined by conditions (I)–(III).

Let us now say some words about solutions of the system (2.19). We have

$$\omega\Phi_{(l)} + k\Phi_{(l)}\Phi + k^3\Phi_{(l)\theta\theta} = \int^{\theta} f_{(l)}(\theta', X, T)\,\mathrm{d}\theta' + \xi_1. \tag{2.23}$$

The solutions of system (2.23) can be easily written in quadratures. Two important cases can be pointed out in our situation:

(I) the function $f_{(l)}(\theta)$ is even, $f_{(l)}(-\theta) = f_{(l)}(\theta)$;

(II) the function $f_{(l)}(\theta)$ is odd, $f_{(l)}(-\theta) = -f_{(l)}(\theta)$.

Proofs of the following two propositions are straightforward.

Proposition 2.1. *For an even smooth periodic $f_{(l)}(\theta)$ the corresponding solution $\Phi_{(l)}(\theta)$ of (2.19) satisfying conditions (2.22) is an odd smooth periodic function.*

Proposition 2.2. *For an odd smooth periodic $f_{(l)}(\theta)$, the corresponding solution $\Phi_{(l)}(\theta)$ of (2.19) satisfying conditions (2.22) is an even smooth periodic function.*

In particular, the function $f_{(1)}(\theta, X, T)$ is given by the following expression:

$$-[\Phi_{(0)T}]^{[1]} - \Phi_{(0)}\Phi_{(0)X} - 3S_X^2\Phi_{(0)\theta\theta X} - 3S_X S_{XX}\Phi_{(0)\theta\theta}.$$

Recall that the mark $^{[1]}$ means that we collect the terms of the degree 1.

Orthogonality conditions (2.21) determine the functions $\sigma_{(1)}(k, \omega, n, k_X, \omega_X, n_X)$ and $\eta_{(1)}(k, \omega, n, k_X, \omega_X, n_X)$. In this way we arrive at the standard Whitham system (2.11) as the zero order approximation of (2.20).

More generally, according to our approach, the functions $f_{(l)}(\theta, X, T)$ will always be represented in the form

$$f_{(l)} = -[\Phi_{(0)T}]^{[l]} + f'_{(l)} = -\Phi_{(0)\omega}\sigma_{(l)} + \Phi_{(0)n}\eta_{(l)} + f'_{(l)},$$

where $f'_{(l)}$ does not contain the terms $\sigma_{(l)}$, $\eta_{(l)}$. The corresponding orthogonality conditions (2.21) recursively determine all the terms $\sigma_{(l)}$, $\eta_{(l)}$.

It is easy to see that the function $f_{(1)}$ is even: $f_{(1)}(-\theta) = f_{(1)}(\theta)$. We obtain therefore that the function $\Phi_{(1)}(\theta)$ is odd $\Phi_{(1)}(-\theta) = -\Phi_{(1)}(\theta)$.

Furthermore, a direct substitution gives

$$f_{(2)} = -[\Phi_{(0)T}]^{[2]} + f'_{(2)} = -\Phi_{(0)\omega}\sigma_{(2)} + \Phi_{(0)n}\eta_{(2)} + f'_{(2)},$$

where $f'_{(2)}(\theta)$ is odd.

Using equations (2.21) for $l = 2$ we immediately get $\sigma_{(2)} \equiv 0$, $\eta_{(2)} \equiv 0$ for the next terms in (2.20). The total function $f_{(2)}(\theta, X, T)$ becomes then an odd function in θ. Hence the second correction $\Phi_{(2)}(\theta)$ is even. By simple induction we obtain the following lemma.

Lemma 2.1. *For the choice of the functions $\Phi(\theta; k, \omega, n)$ corresponding to figure 1 the following statements are true:*

(1) All the even terms $\sigma_{(2l)}(k, \omega, n, \ldots)$, $\eta_{(2l)}(k, \omega, n, \ldots)$ in the deformation of Whitham system (2.20) are identically zero: $\sigma_{(2l)} \equiv 0$, $\eta_{(2l)} \equiv 0$;
(2) All odd corrections $\Phi_{(2l+1)}(\theta, X, T)$, $l \geqslant 0$ in (2.12) are odd functions in θ;
(3) All even corrections $\Phi_{(2l)}(\theta, X, T)$, $l \geqslant 1$ in (2.12) are even functions in θ.

3. Deformation scheme for the case of small amplitude oscillations

The above procedure of deformation has one weak point. Namely, in the procedure described, the higher corrections $\Phi_{(l)}(\theta, X, T)$, as well as the higher deformation terms in system (2.20), are singular in the limit of small amplitude oscillations of $\varphi(x, t)$. The reason for such a singular behavior can be explained in the following way.

Let us rewrite system (2.19) in the form (2.23) i.e.

$$\omega\Phi_{(l)} + k\Phi_{(l)}\Phi + k^3\Phi_{(l)\theta\theta} = g_{(l)}(\theta, X, T),$$

where the right-hand part $g_{(l)}$ given by the expression

$$g_{(l)}(\theta, X, T) \equiv \int_0^\theta f_{(l)}(\theta', X, T)\,\mathrm{d}\theta' + \delta_{(l)}(X, T)$$

is periodic in θ due to conditions (2.21).

We can rewrite this system in the form

$$\hat{Q}_{[X,T]}\Phi_{(l)} = g_{(l)}, \tag{3.1}$$

where

$$\hat{Q}_{[X,T]} \equiv \omega(k, A, n) + k\Phi + k^3\frac{\partial^2}{\partial\theta^2} \tag{3.2}$$

is a self-adjoint operator on the space of 2π-periodic functions.

Operator (3.2) has just one eigenvector Φ_θ with zero eigenvalue on the space of 2π-periodic functions. The constants $\delta_{(l)}(X, T)$ are uniquely determined by the second condition (2.22) for the solutions $\Phi_{(l)}$ of (3.1). It is not difficult to get analytic expressions for $\delta_{(l)}(X, T)$. It is also easy to see that $\delta_{(l)}(X, T) \equiv 0$ for $l = 2s + 1$, $s \geqslant 0$.

Provided that conditions (2.21) are satisfied we can write the solution of (3.1) in the form

$$\Phi_{(l)} = \sum_j \frac{1}{\lambda_j}\xi_j(\theta, X, T)\langle\xi_j, g_{(l)}\rangle, \tag{3.3}$$

where $\xi_j(\theta, X, T)$ are the normalized eigenvectors of $\hat{Q}_{[X,T]}$ corresponding to the non-zero eigenvalues λ_j.

Let us now consider operator (3.2) for the case of small amplitude oscillations:

$$\Phi(\theta, X, T) = n(X, T) + a_0(X, T)\cos\theta + \cdots, \qquad a_0 \to 0.$$

The parameter $a_0(X, T)$ is the amplitude of the first Fourier harmonic of $\Phi(\theta, X, T)$ which is similar to the parameter $A = \Phi_{\max} - \Phi_{\min}$ in the limit $A \to 0$.

Operator (3.2) always has the eigenvector $\xi(\theta, X, T) = \Phi_\theta(\theta, X, T)$ corresponding to zero eigenvalue, which corresponds to the function $-\sin\theta$ in the limit $A \to 0$. The dispersion

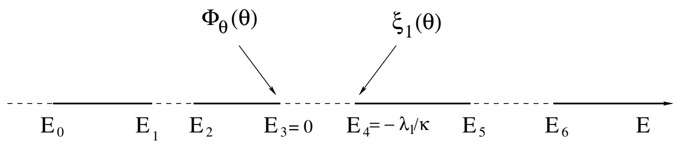

Figure 2. The spectrum of the operator $-\hat{Q}_{[X,T]}/k$. The intervals $[E_0, E_1]$, $[E_2, E_3]$, $[E_4, E_5]$ and $[E_1, E_2]$, $[E_3, E_4]$, $[E_5, E_6]$ represent the energy bands and the energy gaps of a finite size, respectively.

relation $\omega = \omega(k, A, n)$ becomes the dispersion relation of the linear system

$$\omega = -nk + k^3$$

for $A = 0$.

However, the function $\cos\theta$ also gives an eigenvector of the linear operator ($A = 0$) (3.2) corresponding to zero eigenvalue. As a result there exists an eigenvector $\xi_1(\theta, X, T)$ of the operator $\hat{Q}_{[X,T]}$ corresponding to the 'small' eigenvalue $\lambda_1 \to 0$ (for $A \to 0$).

The values $\xi_1(\theta)$ and λ_1 can also be expressed in terms of the elliptic functions in our case. Indeed, if we compare the operator $-\hat{Q}_{[X,T]}/k$ with the Schrödinger operator (2.14) we can easily see that the operator $-\hat{Q}_{[X,T]}/k$ is represented by the Schrödinger operator with a one-zone potential multiplied by 6. It is well known (see for example [5, 4]) that the operator $-\hat{Q}_{[X,T]}/k$ represents a 3-energy gap Schrödinger operator with an elliptic potential in this case.

The spectrum of the operator $-\hat{Q}_{[X,T]}/k$ is shown in figure 2 and we are interested here in the particular functions $\xi_1(\theta)$ and λ_1.

It is not difficult to see then that the eigenfunction $\Phi_\theta(\theta)$ and $\xi_1(\theta)$ correspond to the gap edges $E_3 = 0$ and $E_4 = -\lambda_1/k$. Easy to see also that in the limit of the small amplitude of oscillations, we have $|E_4 - E_3| \to 0$ in the full accordance with the perturbations theory. The expressions for the functions $\xi_1(\theta)$ and λ_1 can be written in the form (see [5])

$$\xi_1(\theta, a, s, \gamma) \sim \mathrm{d}n\left(\frac{K(s)}{\pi}\theta, s\right)\left[1 + 2s^2 - \sqrt{1 - s^2 + 4s^4} - 5s^2\mathrm{sn}^2\left(\frac{K(s)}{\pi}\theta, s\right)\right]$$

$$\begin{aligned}\lambda_1(a, s, \gamma) &= -K^2(s)(2\sqrt{1 - s^2 + 4s^4} - 2 + s^2)k^3/\pi^2 \\ &= -\pi(2\sqrt{1 - s^2 + 4s^4} - 2 + s^2)\left(\frac{a}{6s^2}\right)^{3/2}\Big/ K(s)\end{aligned}$$

in our notations.

By direct substitution it is not difficult to also obtain the following relations for the values of $\omega(k, A, n)$, Φ, $\hat{Q}$, ξ_1 and λ_1:

$$\omega = -kn + k^3 - \frac{a_0^2}{24k} + \mathcal{O}\big(a_0^4\big) \tag{3.4}$$

$$\Phi(\theta, k, A, n) = n + a_0\cos\theta + \frac{a_0^2}{12k^2}\cos 2\theta + \mathcal{O}\big(a_0^3\big) \tag{3.5}$$

$$\hat{Q}_{[k,A,n]} = k^3 - \frac{a_0^2}{24k} + k\,a_0\cos\theta + \frac{a_0^2}{12k}\cos 2\theta + k^3\frac{\partial^2}{\partial\theta^2} + \mathcal{O}\big(a_0^3\big) \tag{3.6}$$

$$\xi_1(\theta, k, A, n) = \cos\theta - \frac{a_0}{2k^2} + \frac{a_0}{6k^2}\cos 2\theta + \mathcal{O}\big(a_0^2\big) \tag{3.7}$$

$$\lambda_1 = -\frac{5a_0^2}{12k} + \mathcal{O}\big(a_0^4\big). \tag{3.8}$$

We can see that the solutions (3.3) become singular in the limit of small amplitude oscillations $A \to 0$ if we do not put additional requirement

$$\langle \xi_1, g_{(l)} \rangle \equiv 0$$

for all $g_{(l)}$. To improve the deformation procedure described above we will use the deformation scheme suggested in [56] for the case of almost linear systems.

Namely, the orthogonality of $g_{(l)}(\theta, X, T)$ to $\xi_1(\theta, X, T)$ can be provided in the following way.

First of all, we choose the parameters (k, A, n) instead of (k, ω, n) as the regular parameters everywhere (including the region $A \to 0$). Now the main approximation in the asymptotic solution (2.12) will again be given by the function $\Phi(S(X, T) + \theta, k, A, n)$ such that $S_X(X, T) = k(X, T)$. So we have again the same approximation with the same relation between S and k as previously at every T. However, we now make also the 'deformation' of time evolution of phase $S(X, T)$ such that $S_T(X, T) \neq \omega(k, A, n)$ anymore. Instead, we now put the deformed relation

$$S_T = \omega(k, A, n) + \sum_{l \geqslant 1} \omega_{(l)}(k, A, n, k_X, A_X, n_X, \ldots) \tag{3.9}$$

connecting the time derivative S_T and the parameters (k, A, n) of the main approximation. Here again all the functions $\omega_{(l)}(k, A, n, k_X, A_X, n_X, \ldots)$ are differential polynomials in $(k_X, A_X, n_X, \ldots)$ of the degree l with coefficients smooth in (k, A, n) according to the same gradation rule, i.e.

(i) all the functions $f(k, A, n)$ have degree 0;
(ii) the derivatives k_{lX}, A_{lX}, n_{lX} have degree l;
(iii) the degree of the product of homogeneous differential polynomials is equal to the sum of their degrees.

As we have already mentioned, the parameter $A = \Phi_{\max} - \Phi_{\min}$ plays here the role of the amplitude of oscillations and we have $A(X, T) \sim a_0(X, T)$ for the small A.

We write now the deformed Whitham system in the form

$$\begin{aligned} k_T &= \left(\omega(k, A, n) + \sum_{l \geqslant 1} \omega_{(l)}(k, A, n, k_X, A_X, n_X, \ldots) \right)_X \\ A_T &= \sum_{l \geqslant 1} \alpha_{(l)}(k, A, n, k_X, A_X, n_X, \ldots) \\ n_T &= \sum_{l \geqslant 1} \eta_{(l)}(k, A, n, k_X, A_X, n_X, \ldots), \end{aligned} \tag{3.10}$$

which gives a full deformation of the Whitham system having a regular behavior in the case of small amplitudes.

The functions $\alpha_{(l)}$, $\eta_{(l)}$ are defined as previously from the orthogonality conditions of the functions $f_{(l)}(\theta, X, T)$ to the 'left' eigenvectors $\Phi(\theta)$ and 1 of the operator $\hat{L}$ corresponding to zero eigenvalues. The functions $\omega_{(l)}$ in (3.9) are defined now from the orthogonality of the functions $g_{(l)}(\theta, X, T)$ to the eigenvector $\xi_1(\theta, X, T)$ of the operator $\hat{Q}_{[X,T]}$ corresponding to the 'small' eigenvalue $\lambda_1(k, A, n)$.

So now we have the condition

$$\int_0^{2\pi} \xi_1(\theta, X, T) g_{(l)}(\theta, X, T) \frac{d\theta}{2\pi} \equiv 0 \tag{3.11}$$

in addition to conditions (2.21). The functions $\lambda_1(k, A, n)$, $\xi_1(\theta, k, A, n)$ are defined by continuity on the whole family of one-phase solutions so we can define the system (3.10) on the full space of parameters.

For our choice of the functions $\Phi(\theta, k, A, n)$, it is easy to prove that the function $\xi_1(\theta, k, A, n)$ is even in θ.

For the solutions $\Phi_{(l)}(\theta, X, T)$ we will have automatically

$$\int_0^{2\pi} \xi_1(\theta, X, T)\Phi_{(l)}(\theta, X, T)\frac{\mathrm{d}\theta}{2\pi} \equiv 0 \tag{3.12}$$

in addition to normalization conditions (2.22).

In the same way as previously the following lemma can be proved for systems (3.9)–(3.10) and the asymptotic expansion

$$\phi(\theta, X, T) = \Phi(S(X, T) + \theta, k, A, n) + \sum_{l\geqslant 1}\Phi_{(l)}(S(X, T) + \theta, X, T). \tag{3.13}$$

Lemma 3.1. *For the 'unified' choice of the functions $\Phi(\theta, k, A, n)$ corresponding to figure 1 the following statements are true.*

(1) All even terms $\sigma_{(2l)}(k, A, n, \ldots)$, $\eta_{(2l)}(k, A, n, \ldots)$ in the deformation of the Whitham system (3.10) are identically zero: $\alpha_{(2l)} \equiv 0$, $\eta_{(2l)} \equiv 0$.

(2) All odd terms $\omega_{(2l+1)}(k, A, n, \ldots)$, $l \geqslant 0$, in the deformation (3.9) of the dispersion relation are identically zero: $\omega_{(2l+1)} \equiv 0$.

(3) All odd corrections $\Phi_{(2l+1)}(\theta, X, T)$, $l \geqslant 0$, in (2.12) are odd in θ.

(4) All even corrections $\Phi_{(2l)}(\theta, X, T)$, $l \geqslant 1$, in (2.12) are even in θ.

So we can rewrite relation (3.9) and the system (3.10) in the form

$$\begin{aligned}
S_T &= \omega(k, A, n) + \sum_{l\geqslant 1}\omega_{(2l)}(k, A, n, k_X, A_X, n_X, \ldots)\\
k_T &= \left(\omega(k, A, n) + \sum_{l\geqslant 1}\omega_{(2l)}(k, A, n, k_X, A_X, n_X, \ldots)\right)_X\\
A_T &= \sum_{l\geqslant 0}\alpha_{(2l+1)}(k, A, n, k_X, A_X, n_X, \ldots)\\
n_T &= \sum_{l\geqslant 0}\eta_{(2l+1)}(k, A, n, k_X, A_X, n_X, \ldots).
\end{aligned} \tag{3.14}$$

We can see that for our choice of the functions $\Phi(\theta; k, A, n)$ the full deformation (3.14) of the Whitham system includes only odd degrees of the expansion in higher derivatives which emphasizes the dispersive character of the deformation.

To calculate the terms $\omega_{(2)}, \alpha_{(3)}, \eta_{(3)}$ let us write down the expressions for the discrepancies $f_{(1)}, f_{(2)}, f_{(3)}$ in the form

$$\begin{aligned}
-f_{(1)} &= \Phi_T^{[1]} + \Phi\Phi_X + 3S_X S_{XX}\Phi_{\theta\theta} + 3S_X^2\Phi_{\theta\theta X} + S_T^{[1]}\Phi_\theta\\
-f_{(2)} &= \Phi_T^{[2]} + \Phi_{(1)T}^{[2]} + 3S_{XX}\Phi_{\theta X} + 3S_X\Phi_{\theta XX} + S_{XXX}\Phi_\theta + S_X\Phi_{(1)}\Phi_{(1)\theta}\\
&\quad + \Phi\Phi_{(1)X} + \Phi_{(1)}\Phi_X + 3S_X S_{XX}\Phi_{(1)\theta\theta} + 3S_X^2\Phi_{(1)\theta\theta X} + S_T^{[2]}\Phi_\theta\\
-f_{(3)} &= \Phi_T^{[3]} + \Phi_{(1)T}^{[3]} + \Phi_{(2)T}^{[3]} + \Phi_{XXX} + 3S_{XX}\Phi_{(1)\theta X} + 3S_X\Phi_{(1)\theta XX} + S_{XXX}\Phi_{(1)\theta}\\
&\quad + \Phi_{(1)}\Phi_{(1)X} + \Phi\Phi_{(2)X} + \Phi_{(2)}\Phi_X + 3S_X S_{XX}\Phi_{(2)\theta\theta} + 3S_X^2\Phi_{(2)\theta\theta X}\\
&\quad + S_X\Phi_{(1)}\Phi_{(2)\theta} + S_X\Phi_{(2)}\Phi_{(1)\theta} + S_T^{[3]}\Phi_\theta.
\end{aligned}$$

Let us recall again that we do not prescribe any certain degree to the operator $\partial/\partial T$, so we have

$$\begin{aligned} S_T &= \omega(k, A, n) + \omega_{(1)}(k, A, n, k_X, A_X, n_X) + \cdots \\ \Phi_T &= \Phi_T^{[1]} + \Phi_T^{[2]} + \Phi_T^{[3]} + \cdots \\ \Phi_{(1)T} &= \Phi_{(1)T}^{[2]} + \Phi_{(1)T}^{[3]} + \Phi_{(1)T}^{[4]} + \cdots \\ \Phi_{(2)T} &= \Phi_{(2)T}^{[3]} + \Phi_{(2)T}^{[4]} + \Phi_{(2)T}^{[5]} + \cdots, \end{aligned}$$

where all $\Phi_{(l)T}^{[s]}$ are differential polynomials in $(k_X, A_X, n_X, \ldots)$ of the degree s with smooth coefficients depending on (k, A, n).

It is easy to see that the only odd in θ term in $f_{(1)}$ is $-S_T^{[1]}\Phi_\theta$. So from the orthogonality of $g_{(1)}$ to $\xi_1(\theta, X, T)$ we get immediately $\omega_{(1)} \equiv S_T^{[1]} \equiv 0$ in accordance to lemma 2. In the same way we also put $\omega_{(l)} \equiv 0$, $\delta_{(l)} \equiv 0$ for all $l = 2s + 1$, $s \geqslant 0$, such that only $\omega_{(2s)}$, $\delta_{(2s)}$ should be computed for $s \geqslant 1$.

We will not need to calculate completely the system (3.14) in this paper; however, let us briefly describe here the scheme for the determination of the functions $\alpha_{(1)}$, $\eta_{(1)}$, $\alpha_{(3)}$, $\eta_{(3)}$, $\omega_{(2)}$. We have

$$\Phi_T^{[1]} = \Phi_a A_T^{[1]} + \Phi_n n_T^{[1]} + \Phi_k k_T^{[1]} = \Phi_a \alpha_{(1)} + \Phi_n \eta_{(1)} + \Phi_k \left(\omega(k, a, n)\right)_X$$

so the orthogonality of $f_{(1)}(\theta, X, T)$ to the functions $\Phi(\theta, X, T)$ and 1 gives the usual expression for $\alpha_{(1)}$, $\eta_{(1)}$ given by the standard system of Whitham. The only even term in $f_{(2)}$ is $-\Phi_T^{[2]}$. So we immediately get $\alpha_{(2)} \equiv 0$, $\eta_{(2)} \equiv 0$ from the orthogonality of $f_{(2)}$ to $\Phi(\theta, X, T)$ and 1. The term $-\Phi_{(1)T}^{[2]}$ is given by

$$-\int \left(\frac{\delta\Phi_{(1)}(X)}{\delta A(Y)} \alpha_{(1)}(Y) + \frac{\delta\Phi_{(1)}(X)}{\delta n(Y)} \eta_{(1)}(Y) + \frac{\delta\Phi_{(1)}(X)}{\delta k(Y)} \omega_Y(Y) \right) \mathrm{d}Y$$

and it is a known function. From the orthogonality of $g_{(2)}$ to $\xi_1(\theta, X, T)$ we get a relation for $\omega_{(2)}(k, A, n, \ldots)$:

$$\omega_{(2)} \int_0^{2\pi} \xi_1(\theta)\left(\Phi(\theta) - \Phi(0)\right) \frac{\mathrm{d}\theta}{2\pi} = \int_0^{2\pi} \xi_1(\theta) g'_{(2)}(\theta) \frac{\mathrm{d}\theta}{2\pi},$$

where

$$\begin{aligned} g'_{(2)} = -\int_0^\theta &\left[\Phi_{(1)T}^{[2]} + 3k_X\Phi_{\theta' X} + 3k\Phi_{\theta' XX} + k_{XX}\Phi_{\theta'} + k\Phi_{(1)}\Phi_{(1)\theta'} + \Phi\Phi_{(1)X}\right. \\ &\left. + \Phi_{(1)}\Phi_X + 3kk_X\Phi_{(1)\theta'\theta'} + 3k^2\Phi_{(1)\theta'\theta' X}\right] \mathrm{d}\theta' + \delta_{(2)}(X, T). \end{aligned}$$

It is convenient to determine the values of $\omega_{(2)}$ and $\delta_{(2)}$ simultaneously from the orthogonality of $g_{(2)}(\theta)$ to both the vectors $\xi_1(\theta)$ and

$$\xi_{\mathrm{tot}}(\theta) = \sum_{s \geqslant 1} \frac{1}{\lambda_{2s}} J_{2s}\xi_{2s}, \qquad J_{2s} = \int_0^{2\pi} \xi_{2s}(\theta) \frac{\mathrm{d}\theta}{2\pi}$$

and add the relation

$$\omega_{(2)} \int_0^{2\pi} \xi_{\mathrm{tot}}(\theta)\left(\Phi(\theta) - \Phi(0)\right) \frac{\mathrm{d}\theta}{2\pi} = \int_0^{2\pi} \xi_{\mathrm{tot}}(\theta) g'_{(2)}(\theta) \frac{\mathrm{d}\theta}{2\pi},$$

which gives a non-degenerate linear system on the values $\omega_{(2)}$, $\delta_{(2)}$.[1]

[1] Both the functions $\xi_1(\theta)$, $\xi_{\mathrm{tot}}(\theta)$ have in fact explicit expressions in terms of elliptic functions.

Repeating all the arguments we get $\omega_{(3)} \equiv S_T^{[3]} \equiv 0$ from the orthogonality of $g_{(3)}$ to $\xi_1(\theta, X, T)$. We also have $\Phi_{(1)T}^{[3]} \equiv 0$, $A_T^{[2]} \equiv 0$, $n_T^{[2]} \equiv 0$, $k_T^{[2]} \equiv 0$. The function $-\Phi_{(2)T}^{[3]}$ is given by

$$-\int \left(\frac{\delta \Phi_{(2)}(X)}{\delta A(Y)} \alpha_{(1)}(Y) + \frac{\delta \Phi_{(2)}(X)}{\delta n(Y)} \eta_{(1)}(Y) + \frac{\delta \Phi_{(2)}(X)}{\delta k(Y)} \omega_Y(Y) \right) \mathrm{d}Y$$

and is a known function again.

We have then

$$\Phi_T^{[3]} = \Phi_A \alpha_{(3)} + \Phi_n \eta_{(3)} + \Phi_k (\omega_{(2)})_X$$

so we obtain the functions $\alpha_{(3)}, \eta_{(3)}$ from the orthogonality of $f_{(3)}$ to $\Phi(\theta, X, T)$ and 1.

It is also easy to see that the procedure can be extended to any order l such that all the terms $\omega_{(2l)}, \alpha_{(2l+1)}, \eta_{(2l+1)}$ will be uniquely determined.

The system (3.14) determines the evolution of the parameters (k, A, n) of the zero approximation of (2.12) such that the following conditions are satisfied.

(I′) All the functions $\Phi(\theta; k, A, n)$ are chosen in the way shown in figure 1.

(II′) The modulated phase $S(X, T)$ and the parameters (k, A, n) of the zero approximation are connected by the relation

$$S_X(X, T) = k(X, T).$$

(III′) The higher corrections $\Phi_{(l)}(\theta, X, T), l \geqslant 1$, satisfy normalization conditions (2.22) and (3.12).

We would like to introduce a small parameter ϵ according to our gradation rule for more convenient notations. System (3.14) will be rewritten in the form

$$S_T = \omega(k, A, n) + \sum_{l \geqslant 1} \epsilon^{2l} \omega_{(2l)}(k, A, n, k_X, A_X, n_X, \ldots) \tag{3.15}$$

$$\begin{aligned} k_T &= \left(\omega(k, A, n) + \sum_{l \geqslant 1} \epsilon^{2l} \omega_{(2l)}(k, A, n, k_X, A_X, n_X, \ldots) \right)_X \\ A_T &= \sum_{l \geqslant 0} \epsilon^{2l} \alpha_{(2l+1)}(k, A, n, k_X, A_X, n_X, \ldots) \\ n_T &= \sum_{l \geqslant 0} \epsilon^{2l} \eta_{(2l+1)}(k, A, n, k_X, A_X, n_X, \ldots). \end{aligned} \tag{3.16}$$

The asymptotic expansion (3.13) will also be rewritten in the form

$$\phi(\theta, X, T) = \Phi\left(\frac{S(X, T)}{\epsilon} + \theta, k, A, n \right) + \sum_{l \geqslant 1} \epsilon^l \Phi_{(l)} \left(\frac{S(X, T)}{\epsilon} + \theta, X, T \right) \tag{3.17}$$

according to the gradation rules for the functions $S(X, T)$ and $\Phi_{(l)}(\theta, X, T)$.

In these new notations we can actually see that system (3.22)–(3.23) describes the asymptotic solutions of the equation

$$\varphi_T + \varphi \varphi_X + \epsilon^2 \varphi_{XXX} = 0, \tag{3.18}$$

where the small dispersion ϵ^2 arises after the rescaling $T \to \epsilon T$, $X \to \epsilon X$. So our further considerations will be applied to the KdV equation in the small-dispersion form (3.18).

Let us emphasize, however, that (3.17) is not an ϵ-expansion of the asymptotic solution of (3.18) since all of the functions $k(X, T, \epsilon)$, $A(X, T, \epsilon)$, $n(X, T, \epsilon)$ are solutions of the

ϵ-dependent system (3.16), such that expansion (3.17) can contain more complicated ϵ-dependence. According to our rules we should not separate the different orders in ϵ of the functions $k(X, T, \epsilon)$, $A(X, T, \epsilon)$, $n(X, T, \epsilon)$ and only use the gradation rules formulated above for the ϵ-dependent functions[2].

The solutions of system (3.16) can be considered in different ways. Thus, it is easy to define the formal graded form of solutions of (3.16):

$$k(X, T) = k(X, 0) + \sum_{l \geqslant 1} T^l \epsilon^{l-1} K_{(l)}\ (k(X, 0), A(X, 0), n(X, 0), \ldots)$$

$$A(X, T) = A(X, 0) + \sum_{l \geqslant 1} T^l \epsilon^{l-1} A_{(l)}\ (k(X, 0), A(X, 0), n(X, 0), \ldots)$$

$$n(X, T) = n(X, 0) + \sum_{l \geqslant 1} T^l \epsilon^{l-1} N_{(l)}\ (k(X, 0), A(X, 0), n(X, 0), \ldots)$$

for $0 < T < \delta$, where all $K_{(l)}$, $A_{(l)}$, $N_{(l)}$ are local functionals of $k(X, 0)$, $A(X, 0)$, $n(X, 0)$ and their derivatives having the corresponding degree.

However, a more complicated treatment of the solutions of (3.16) connected with their global behavior based on the so-called quasitriviality transformations (see [19, 20]) of parameters (k, A, n) is also possible and seems to be very important in the theory of the deformed Whitham systems.

Let us also recall that the KdV equation (2.8) has an infinite series of conservation laws which can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{P}^\nu(\varphi, \varphi_x, \ldots) = \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{Q}^\nu(\varphi, \varphi_x, \ldots) \qquad (3.19)$$

with some local functionals $\mathcal{P}^\nu(\varphi, \varphi_x, \ldots)$, $\mathcal{Q}^\nu(\varphi, \varphi_x, \ldots)$. For equation (3.18), the corresponding relations can be written respectively

$$\frac{\mathrm{d}}{\mathrm{d}T} \mathcal{P}^\nu(\varphi, \epsilon\varphi_X, \ldots) = \frac{\mathrm{d}}{\mathrm{d}X} \mathcal{Q}^\nu(\varphi, \epsilon\varphi_X, \ldots), \qquad \nu = 0, 1, 2, \ldots. \qquad (3.20)$$

According to standard numeration we put

$$\mathcal{P}^0 = \varphi, \qquad \mathcal{Q}^0 = -\tfrac{1}{2}\varphi^2 - \epsilon^2 \varphi_{XX}$$

for $\nu = 0$ and we have the conservation of the Casimir function

$$N = \int_{-\infty}^{+\infty} \varphi \, \mathrm{d}X$$

for the Gardner–Zakharov–Faddeev bracket in this case.

For $\nu = 1$ it is put traditionally

$$\mathcal{P}^1 = \tfrac{1}{2}\varphi^2, \qquad \mathcal{Q}^1 = -\tfrac{1}{3}\varphi^3 - \epsilon^2 \varphi\varphi_{XX} + \tfrac{1}{2}\epsilon^2 \varphi_X^2,$$

which corresponds to the conservation of the momentum functional for the same bracket.

For $\nu = 2$ we put

$$\mathcal{P}^2 = \tfrac{1}{6}\varphi^3 - \tfrac{1}{2}\epsilon^2 \varphi_X^2, \qquad \mathcal{Q}^2 = -\tfrac{1}{8}\varphi^4 - \tfrac{1}{2}\epsilon^2 \varphi^2 \varphi_{XX} + \epsilon^2 \varphi \varphi_X^2 + \epsilon^4 \varphi_X \varphi_{XXX} - \tfrac{1}{2}\epsilon^4 \varphi_{XX}^2,$$

which gives the conservation of energy in the Gardner–Zakharov–Faddeev Poisson structure.

The higher conservation laws are connected with the integrable nature of the KdV equation and arise from the method of the inverse scattering problem.

[2] As it was pointed out in [55], the series (3.17) (or (3.13)) corresponds to the expansion with respect to X-derivatives of the 'renormalized' ϵ-dependent parameters of the main approximation $\Phi(\theta, k, A, n)$ which gives these specific rules of constructing of the series (3.17).

It is not difficult to see that the conservation laws of the KdV equation (3.18) give conservation laws for system (3.16) after the 'averaging' of the corresponding densities $\mathcal{P}^\nu$, $\mathcal{Q}^\nu$ on the asymptotic family (3.17). Indeed, after the substitution of (3.17) into (3.20) and integration w.r.t. θ, we get the relations

$$\frac{\mathrm{d}}{\mathrm{d}T}\langle\mathcal{P}^\nu\rangle = \frac{\mathrm{d}}{\mathrm{d}X}\langle\mathcal{Q}^\nu\rangle, \tag{3.21}$$

where the quantities $\langle\mathcal{P}^\nu\rangle$, $\langle\mathcal{Q}^\nu\rangle$ are given by the substitution of solutions (3.17) in the expressions for $\mathcal{P}^\nu$ and $\mathcal{Q}^\nu$ and integration w.r.t. θ over the period. It is also not difficult to see that the values $\langle\mathcal{P}^\nu\rangle$, $\langle\mathcal{Q}^\nu\rangle$ are expressed in this case as the local functionals of the parameters (k, A, n) and their X-derivatives

$$\langle\mathcal{P}^\nu\rangle = \langle\mathcal{P}^\nu\rangle(k, A, n, k_X, A_X, n_X, \ldots), \qquad \langle\mathcal{Q}^\nu\rangle = \langle\mathcal{Q}^\nu\rangle(k, A, n, k_X, A_X, n_X, \ldots),$$

which are polynomial in the derivatives of (k, A, n) and can be written in the graded form we introduced above. Relations (3.21) give then an infinite set of conservation laws for system (3.16) written in the same graded form.

We can also see that the values of independent integrals $\langle\mathcal{P}^\nu\rangle$ can also be chosen as the parameters of solutions (3.17) such that the values (k, A, n) will be expressed in the form of graded expansions with respect to the X-derivatives of $\langle\mathcal{P}^\nu\rangle$ given by the 'inversion' of the corresponding expansions for $\langle\mathcal{P}^\nu\rangle$. System (3.16), written in the corresponding parameters (say $\langle\mathcal{P}^0\rangle$, $\langle\mathcal{P}^1\rangle$, $\langle\mathcal{P}^2\rangle$), has then a conservative form and expresses the balance of the chosen conservation laws.

Remark 3.1. Let us come back now to figure 2 representing the spectrum of the operator $-\hat{Q}_{[X,T]}/k$. Let us consider the limit $k \to 0$ now and consider the spectrum of $-\hat{Q}_{[X,T]}/k$ on the space of 2π-periodic functions in this limit. We can say, first of all, that the sizes of the energy bands $[E_0, E_1]$, $[E_2, E_3]$, $[E_4, E_5]$ tend to zero in this situation giving the 'splitting' of the three localized quantum states of the corresponding decreasing (one-soliton) potential arising at $k \to 0$.[3] The potential $\Phi(kx; k, A, n)$ represents in this case a 'lattice' of distant one-soliton solutions with the period $\sim k^{-1}$ with $k \to 0$.

Let us note now that the eigenvalues of the operator $-\hat{Q}_{[X,T]}/k$ for the energies $E > E_6$ are double degenerated on the space of periodic functions and represent the 'boundaries of the gaps of zero width' in the spectrum of $-\hat{Q}_{[X,T]}/k$. It is not difficult to see also that the distance between these eigenvalues decreases ($\sim k^2$) in the limit $k \to 0$. Moreover, the size of the gap $[E_5, E_6]$ (as well as the energy band $[E_4, E_5]$) decreases also in this situation. As a result, we can see that another instability arises in our scheme for the case of the small k due to the large number of 'small' eigenvalues of $\hat{Q}_{[X,T]}$ in this limit. This fact means most probably that the averaging methods are not very applicable in the limit $k \to 0$ where the 'multi-soliton' description seems to give more adequate picture.

At last let us say that we believe that it is enough to keep only the first (ϵ^2) dispersive terms in system (3.16) for the description of many oscillating regimes arising in the KdV theory. Finally, we arrive at the system

$$S_T = \omega(k, A, n) + \epsilon^2\omega_{(2)}(k, A, n, \ldots) \tag{3.22}$$

$$\begin{aligned} k_T &= (\omega(k, A, n) + \epsilon^2\omega_{(2)}(k, A, n, \ldots))_X \\ A_T &= a_{(1)}(k, A, n, k_X, A_X, n_X) + \epsilon^2 a_{(3)}(k, A, n, \ldots) \\ n_T &= \eta_{(1)}(k, A, n, k_X, A_X, n_X) + \epsilon^2\eta_{(3)}(k, A, n, \ldots) \end{aligned} \tag{3.23}$$

[3] The limit $k \to 0$ of a one-phase solution of KdV gives a one-soliton solution corresponding to a reflectionless potential with one localized quantum state for the Lax operator (2.14). The same solution gives a potential with three bounded states (one with $E = 0$) for the operator $\hat{Q}_{[X,T]}$ given by (3.2) in the same limit.

since we believe that it already demonstrates many essential features of the full system (3.15) and (3.16).

System (3.23) should be considered as a system of differential equations in the ordinary sense, in particular, all the solutions of (3.23) are supposed to be well-defined functions of X and T with some concrete behavior depending on the regime under investigation.

In the following sections we consider the questions connected with the Hamiltonian structures of system (3.16) which is the main subject of this paper. So, we will now consider the initial KdV equation as a part of an integrable hierarchy having two local Hamiltonian structures and discuss a possibility of the 'averaging' of the Hamiltonian structures to obtain the Hamiltonian structures of the Dubrovin–Zhang type for system (3.16).

4. The commuting flows and the Hamiltonian structures

It is well known that the KdV equation represents the first nontrivial flow of the integrable KdV hierarchy generated by the higher KdV integrals

$$I^{\nu} = \int_{-\infty}^{+\infty} \mathcal{P}^{\nu}(\varphi, \varphi_x, \ldots)\, \mathrm{d}x$$

with respect to the Gardner–Zakharov–Faddeev bracket

$$\{\varphi(x), \varphi(y)\} = \delta'(x-y)$$

or the Magri bracket

$$\{\varphi(x), \varphi(y)\} = \delta'''(x-y) + \tfrac{2}{3}\varphi(x)\delta'(x-y) + \tfrac{1}{3}\varphi_x\delta(x-y).$$

All the higher KdV flows have the similar form

$$\varphi_{t^{\nu}} = f^{\nu}(\varphi, \varphi_x, \varphi_{xx}, \ldots) \tag{4.1}$$

and give an infinite set of commuting integrable flows.

The commuting flows (4.1) can also be written in the 'small dispersion' form

$$\epsilon\varphi_{T^{\nu}} = f^{\nu}(\varphi, \epsilon\varphi_X, \epsilon^2\varphi_{XX}, \ldots), \tag{4.2}$$

which gives the commuting flows for the KdV equation written in the form (3.18).

It is natural to expect then that the higher flows (4.2) of the KdV hierarchy generate the commuting flows for the deformed Whitham system (3.16) such that we get an 'integrable' hierarchy starting from system (3.16) on the 'averaged' level.

We have to introduce now the 'extended functional space' $\mathcal{M} = \{\varphi(\theta, X)\}$ consisting of smooth functions $\varphi(\theta, X)$ which are 2π-periodic in θ at every X. For our further purposes we need to introduce also a 'submanifold' $\mathcal{K} \in \mathcal{M}$ corresponding to the set of solutions (3.17) which will play the basic role in our considerations. Let us note here that all our considerations will be connected with the formal asymptotic series in the derivatives of parameters of one-phase solutions of KdV, so we also define the submanifold $\mathcal{K}$ in the same form, i.e. as a formal submanifold having the asymptotic sense.

Thus, we define the submanifold $\mathcal{K}$ in the space of functions $\varphi(\theta, X)$ by the following rule.

(1) The function $\varphi(\theta, X)$ belongs to the family $\mathcal{K}$ if it represents one of solutions (3.17), i.e.

$$\varphi(\theta, X) = \Phi\left(\frac{S(X)}{\epsilon} + \theta, k, A, n\right) + \sum_{l\geqslant 1} \epsilon^l \Phi_{(l)}\left(\frac{S(X)}{\epsilon} + \theta, [k, A, n], X\right)$$

with some functions $(S(X), A(X), n(X))$, where $k(X) = S_X(X)$.

(2) We put the following relation between the functions $S(X)$ and $k(X)$[4]:

$$S(X) = \frac{1}{2} \int_{-\infty}^{+\infty} \operatorname{sgn}(X - Y) k(Y) \, \mathrm{d}Y \tag{4.3}$$

The functions $(k(X), A(X), n(X))$ play the role of 'coordinates' on the submanifold $\mathcal{K}$, so we consider $\mathcal{K}$ a manifold parametrized by three functional parameters.

Let us formulate here the theorem which connects the higher flows (4.2) with the commuting flows of system (3.16).

Theorem 4.1. *Every higher KdV flow (4.2) leaves invariant the family of formal solutions (3.17) and generates a commuting flow for the deformed Whitham system (3.16) which can be represented in the same graded form*

$$S_{T^\nu} = \omega^\nu(k, A, n) + \sum_{l \geqslant 1} \epsilon^{2l} \omega^\nu_{(2l)}(k, A, n, k_X, A_X, n_X, \ldots) \tag{4.4}$$

$$\begin{aligned} k_{T^\nu} &= \left(\omega^\nu(k, A, n) + \sum_{l \geqslant 1} \epsilon^{2l} \omega^\nu_{(2l)}(k, A, n, k_X, A_X, n_X, \ldots) \right)_X \\ A_{T^\nu} &= \sum_{l \geqslant 0} \epsilon^{2l} \alpha^\nu_{(2l+1)}(k, A, n, k_X, A_X, n_X, \ldots) \\ n_{T^\nu} &= \sum_{l \geqslant 0} \epsilon^{2l} \, \eta^\nu_{(2l+1)}(k, A, n, k_X, A_X, n_X, \ldots) \end{aligned} \tag{4.5}$$

as system (3.16).

Proof. Let us consider the formal asymptotic series

$$\varphi\,(\theta, X, T, T^\nu) = \sum_{l \geqslant 0} \epsilon^l \, \Psi_{(l)} \left(\frac{S(X, T)}{\epsilon} + \theta, X, T, T^\nu \right), \tag{4.6}$$

where every function $\Psi_{(l)}(\theta, X, T, T^\nu)$ is a local functional of $k_0(X, T) = S_{0X}(X, T)$, $A_0(X, T)$, $n_0(X, T)$ and their X-derivatives which is polynomial in derivatives and has degree l according to the gradation rule we introduced above. We require that series (4.6) coincides with asymptotic series (3.17) with the same parameters $k_0(X, T)$, $A_0(X, T)$, $n_0(X, T)$ for $T^\nu = 0$:

$$\Psi_{(0)}(\theta, X, T, 0) = \Phi(\theta, k_0, A_0, n_0), \qquad \Psi_{(l)}(\theta, X, T, 0) = \Phi_{(l)}(\theta, X, T)$$

and satisfies the higher KdV equation (4.2) for $T^\nu > 0$.

After the substitution of (4.6) into (4.2) in the graded form we get a chain of evolution equations on the functions $\Psi_{(l)}(\theta, X, T, T^\nu)$ at every degree l:

$$\frac{\mathrm{d}}{\mathrm{d}T^\nu} \Psi_{(l)}(\theta, X, T, T^\nu) = \Lambda_{(l)}(\boldsymbol{\Psi}, \boldsymbol{\Psi}_\theta, \boldsymbol{\Psi}_X, \ldots, k_0, A_0, n_0, k_{0X}, A_{0X}, n_{0X}, \ldots), \tag{4.7}$$

where every $\Lambda_{(l)}$ depends only on $\Psi_{(s)}$ with $s \leqslant l$.

It is not difficult to check the following relations for $T^\nu = 0$:

$$\begin{aligned} \Psi_{(l)}(-\theta, X, T, 0) &= (-1)^l \, \Psi_{(l)}(\theta, X, T, 0), \\ \Lambda_{(l)}(-\theta, X, T, 0) &= (-1)^{l+1} \Lambda_{(l)}(\theta, X, T, 0), \end{aligned} \tag{4.8}$$

where $l \geqslant 0$.

[4] Let us assume here that the relations $k(X) \to 0$ for $X \to \pm\infty$ are imposed. However, the procedure will give us a local deformed Poisson bracket on the space $(k(X), A(X), n(X))$, so this condition will not be important in fact for the final result.

Let us assume for simplicity that all the systems (4.7) have smooth solutions on some interval $T^\nu \in [0, \delta]$ with our initial data, such that we get a unique formal series (4.6) satisfying our requirements on the same interval. Since equation (4.2) gives a commuting flow for the KdV equation (3.18) we get that series (4.6) gives a formal solution of (3.18) at every $T^\nu \in [0, \delta]$. However, series (4.6) cannot be considered as the asymptotic series (3.17) for $T^\nu > 0$ since the normalization conditions (2.22), (3.12) will be in general destroyed by the evolution systems (4.7)[5].

Nonetheless, series (4.6) can be represented in form (3.17) after a redefinition of parameters

$$(k_0(X,T), A_0(X,T), n_0(X,T)) \to (k(X,T,T^\nu), A(X,T,T^\nu), n(X,T,T^\nu)) \tag{4.9}$$

and a re-expansion of (4.6) in the new graded form. It is convenient then to represent the redefinition of (k, A, n) in the differential graded form (4.5) which gives the required evolution system on family (3.17).

Let us discuss finally the possibility of constructing the required system (4.5) on the space of parameters $(k(X), A(X), n(X))$ which will prove the theorem. Indeed, the function $n(X, T, T^\nu)$ is given by the integral

$$\int_0^{2\pi} \varphi(\theta, X, T, T^\nu) \frac{\mathrm{d}\theta}{2\pi}$$

according to the definition, so we immediately get the graded equation

$$\frac{\mathrm{d}n}{\mathrm{d}T^\nu} = \sum_{l=0}^{\infty} \int_0^{2\pi} \Lambda_{(l)}(\theta, X, T, T^\nu) \frac{\mathrm{d}\theta}{2\pi},$$

which makes satisfied the second relation (2.22) for $T^\nu > 0$.

Let us consider now the first relation (2.22) and relation (3.12). We have to find now two more functions $S(X, T, T^\nu)$, $A(X, T, T^\nu)$ such that the function $\Phi(\theta, S_X, A, n)$ satisfies the conditions

$$\int_0^{2\pi} \Phi_\theta\left(\frac{S(X,T,T^\nu)}{\epsilon} + \theta, S_X, A, n\right) \sum_{l\geqslant 0} \epsilon^l \Psi_{(l)}\left(\frac{S(X,T)}{\epsilon} + \theta, X, T, T^\nu\right) \frac{\mathrm{d}\theta}{2\pi} \equiv 0$$

$$\int_0^{2\pi} \xi_1\left(\frac{S(X,T,T^\nu)}{\epsilon} + \theta, S_X, A, n\right) \left(\sum_{l\geqslant 0} \epsilon^l \Psi_{(l)}\left(\frac{S(X,T)}{\epsilon} + \theta, X, T, T^\nu\right)\right.$$
$$\left. - \Phi\left(\frac{S(X,T,T^\nu)}{\epsilon} + \theta, S_X, A, n\right)\right) \frac{\mathrm{d}\theta}{2\pi} \equiv 0$$

to be an appropriate main term in the 're-expanded' series (4.6).

We also assume $S(X,T,0) = S_0(X,T), k(X,T,0) = k_0(X,T), A(X,T,0) = A_0(X,T)$, $n(X,T,0) = n_0(X,T)$ according to our scheme.

Differentiating the first relation with respect to T^ν at $T^\nu = 0$ we get

$$\int_0^{2\pi} \left(\frac{1}{\epsilon} S_{T^\nu} \Phi_{\theta\theta} + k_{T^\nu} \Phi_{\theta k} + A_{T^\nu} \Phi_{\theta A} + n_{T^\nu} \Phi_{\theta n}\right) \sum_{l\geqslant 0} \epsilon^l \Psi_{(l)} \frac{\mathrm{d}\theta}{2\pi}$$
$$+ \int_0^{2\pi} \Phi_\theta \sum_{l\geqslant 0} \epsilon^l \Lambda_{(l)}|_{(T^\nu = 0)} \frac{\mathrm{d}\theta}{2\pi} = 0. \tag{4.10}$$

[5] The function $\Psi_{(0)}(\theta, X, T, T^\nu)$ will remain the one-phase solution for $T^\nu > 0$ in this situation; however, the normalization $\Psi_{(0)\theta}(0, X, T, T^\nu) = 0$ will also be destroyed by the higher KdV flow on the one-phase solutions.

We are going to obtain a graded linear system for the determination of the time derivatives S_{T^ν}, A_{T^ν} in the graded form. Using the facts

$$\Psi_{(0)}(\theta, X, T, 0) = \Phi\left(\frac{S(X,T)}{\epsilon} + \theta, k, A, n\right)$$

$$\Lambda_{(0)}(\theta, X, T, 0) = \frac{\omega^\nu(k, A, n)}{\epsilon}\Phi_\theta\left(\frac{S(X,T)}{\epsilon} + \theta, k, A, n\right),$$

where $\omega^\nu(k, A, n)$ is the frequency corresponding to the flow f^ν on the space of the one-phase solutions of KdV we get from equation (4.10) $S_{T^\nu} = \omega^\nu(k, A, n)$ at $T^\nu = 0$ in the main approximation.

Using also relations (4.8) we can write actually

$$S_{T^\nu} = \omega^\nu(k, A, n) + \mathcal{O}(\epsilon^2)$$

at $T^\nu = 0$ for the derivative S_{T^ν}.

After the differentiation of the second relation w.r.t. T^ν at $T^\nu = 0$, we get the following relation:

$$A_{T^\nu}\int_0^{2\pi}\xi_1\Phi_A\frac{\mathrm{d}\theta}{2\pi} = \int_0^{2\pi}\left(\frac{1}{\epsilon}S_{T^\nu}\xi_{1\theta} + k_{T^\nu}\xi_{1k} + A_{T^\nu}\xi_{1A} + n_{T^\nu}\xi_{1n}\right)\sum_{l\geqslant 1}\epsilon^l\Phi_{(l)}\frac{\mathrm{d}\theta}{2\pi}$$
$$+\int_0^{2\pi}\xi_1\left(\sum_{l\geqslant 0}\epsilon^{2l+1}\Lambda_{(2l+1)}|_{(T^\nu=0)} - k_{T^\nu}\Phi_k - n_{T^\nu}\Phi_n\right)\frac{\mathrm{d}\theta}{2\pi} \tag{4.11}$$

at $T^\nu = 0$.

The function $\int_0^{2\pi}\xi_1\Phi_A\,\mathrm{d}\theta/2\pi$ is a strictly positive function on the space of parameters (k, A, n). Using this fact it is not difficult to see then that the form of the linear system (4.10)–(4.11) defines the unique representation of the derivatives S_{T^ν}, A_{T^ν} at $T^\nu = 0$ in the graded form being uniquely resolvable at every step of the determination of $S^{[s]}_{T^\nu}$ and $A^{[s]}_{T^\nu}$. Using also relations (4.8) it is not difficult to prove that we obtain the 'purely dispersive' system (4.5) in this situation.

For $0 < T^\nu < \delta$ system (4.10)–(4.11) is still resolvable with respect to the derivatives S_{T^ν}, A_{T^ν}, such that we have

$$S_{T^\nu} = \sum_{l\geqslant 0}\epsilon^{2l}\omega^\nu_{(2l)}(T^\nu, k_0, A_0, n_0, k_{0X}, A_{0X}, n_{0X}, \ldots)$$

$$k_{T^\nu} = \left(\sum_{l\geqslant 0}\epsilon^{2l}\omega^\nu_{(2l)}(T^\nu, k_0, A_0, n_0, k_{0X}, A_{0X}, n_{0X}, \ldots)\right)_X$$

$$A_{T^\nu} = \sum_{l\geqslant 0}\epsilon^{2l}\alpha^\nu_{(2l+1)}(T^\nu, k_0, A_0, n_0, k_{0X}, A_{0X}, n_{0X}, \ldots)$$

$$n_{T^\nu} = \sum_{l\geqslant 0}\epsilon^{2l}\eta^\nu_{(2l+1)}(T^\nu, k_0, A_0, n_0, k_{0X}, A_{0X}, n_{0X}, \ldots),$$

where all the functions $\omega^\nu_{(2l)}$, $\alpha^\nu_{(2l+1)}$, $\eta^\nu_{(2l+1)}$ become dependent on T^ν and do not coincide with the functions from (4.5) since all the functions $\Psi_{(l)}(\theta, X, T, T^\nu)$ become different from $\Phi_{(l)}(\theta, X, T)$. However, if we represent the solutions of this system, say, in the formal graded form, we will be able to 're-expand' the formal solution (4.6) according to change of parameters (4.9).

Thus we can write now the new formal graded expansion for series (4.6)

$$\varphi(\theta, X, T, T^\nu) = \sum_{l\geqslant 0} \epsilon^l \, \tilde{\Psi}_{(l)} \left(\frac{S(X,T)}{\epsilon} + \theta, X, T, T^\nu \right) \tag{4.12}$$

according to change of parameters of expansion (4.9) at every T^ν. System (4.7) can also be easily rewritten for the functions $\tilde{\Psi}_{(l)}(\theta, X, T, T^\nu)$

$$\frac{\mathrm{d}}{\mathrm{d}T^\nu} \tilde{\Psi}_{(l)}(\theta, X, T, T^\nu) = \tilde{\Lambda}_{(l)}(T^\nu, \tilde{\boldsymbol{\Psi}}, \tilde{\boldsymbol{\Psi}}_\theta, \tilde{\boldsymbol{\Psi}}_X, \ldots, k, A, n, k_X, A_X, n_X, \ldots)$$

using system (4.12) in this situation. As a result, we will get the asymptotic series (4.12) satisfying all conditions (I′)–(III′).

Finally, we get that the asymptotic series (4.12) gives a formal solution of (3.18) satisfying all conditions (I′)–(III′) at $0 < T^\nu < \delta$. As we saw above, solutions (4.12) should coincide in this case with the formal graded solution (3.17) so we get the invariance of the family (3.17) under the higher KdV flows. The evolution of parameters (k, A, n) is ruled then by system (4.5) for all $T^\nu > 0$ and the commutativity of (4.5) with (3.16) follows directly from the commutativity of (4.2) and (3.18).

At last, let us note now that for our conditions $k \to 0$, $X \to \pm\infty$, we also have $S_{T^\nu}(X) \to 0$, $X \to \pm\infty$, which also gives the conservation of condition (4.3) in our situation. □

It is also not difficult to see that system (4.5) coincides with the deformed Whitham system for the higher KdV flow (4.2) defined by the same normalization conditions (I′)–(III′).

Let us discuss now the Hamiltonian properties of system (3.16) following from the Hamiltonian properties of the KdV equation (3.18). According to the general ideology of the deformation of systems of hydrodynamic type, we will assume the existence of Hamiltonian structures for the deformed Whitham system given by the deformations of the Hamiltonian structures of hydrodynamic type, i.e. the Hamiltonian structures having the form

$$\{U^\nu(X), U^\mu(Y)\} = \{U^\nu(X), U^\mu(Y)\}_0 + \sum_{k\geqslant 2} \epsilon^{k-1} \sum_{s=0}^{k} B^{\nu\mu}_{(k)s}(\mathbf{U}, \mathbf{U}_X, \ldots, \mathbf{U}_{(k-s)X}) \, \delta^{(s)}(X - Y), \tag{4.13}$$

where all $B^{\nu\mu}_{(k)s}$ are polynomials w.r.t. derivatives $\mathbf{U}_X, \ldots, \mathbf{U}_{(k-s)X}$ and have degree $(k - s)$.

We call deformations of the Hamiltonian structure of form (4.13) the deformations of Dubrovin–Zhang type. Bracket (4.13) gives a deformation of the local homogeneous bracket of hydrodynamic type (Dubrovin–Novikov bracket) which according to the definition has the following form:

$$\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(\mathbf{U})\delta'(X - Y) + b^{\nu\mu}_\lambda(\mathbf{U}) U^\lambda_X \, \delta(X - Y) \tag{4.14}$$

The corresponding Hamiltonian operator $\hat{J}^{\nu\mu}$ can be written as

$$\hat{J}^{\nu\mu} = g^{\nu\mu}(\mathbf{U}) \frac{\partial}{\partial x} + b^{\nu\mu}_\lambda(\mathbf{U}) U^\lambda_X$$

and is homogeneous w.r.t. transformation $X \to aX$.

Every functional H of hydrodynamic type, i.e. the functional having the form

$$H = \int_{-\infty}^{+\infty} h(\mathbf{U}) \, \mathrm{d}X$$

generates the system of hydrodynamic type

$$U^\nu_T = V^\nu_\mu(\mathbf{U}) U^\mu_X, \qquad \nu, \mu = 1, \ldots, N, \tag{4.15}$$

where $V^{\nu}_{\mu}(U)$ is some $N \times N$ matrix depending on the variables $U^1, \ldots, U^N$ according to the formula

$$U^{\nu}_{T} = \hat{J}^{\nu\mu} \frac{\delta H}{\delta U^{\mu}(X)} = g^{\nu\mu}(\mathbf{U}) \frac{\partial}{\partial x} \frac{\partial h}{\partial U^{\mu}} + b^{\nu\mu}_{\lambda}(\mathbf{U}) \frac{\partial h}{\partial U^{\mu}} U^{\lambda}_{X}. \tag{4.16}$$

The DN-bracket (4.14) is called non-degenerate if $\det \| g^{\nu\mu}(\mathbf{U}) \| \neq 0$.

As was shown by Dubrovin and Novikov, the theory of DN-brackets is closely connected with Riemannian geometry [11–13]. In fact, it follows from the skew-symmetry of (4.14) that the coefficients $g^{\nu\mu}(\mathbf{U})$ give in the non-degenerate case the contravariant pseudo-Riemannian metric on the manifold $\mathcal{M}^N$ with coordinates $(U^1, \ldots, U^N)$ while the functions $\Gamma^{\nu}_{\mu\lambda}(\mathbf{U}) = -g_{\mu\alpha}(\mathbf{U}) b^{\alpha\nu}_{\lambda}(\mathbf{U})$ (where $g_{\nu\mu}(\mathbf{U})$ is the corresponding metric with lower indices) give the connection coefficients compatible with metric $g_{\nu\mu}(\mathbf{U})$. The validity of Jacobi identity requires then that $g_{\nu\mu}(\mathbf{U})$ is actually a flat metric on the manifold $\mathcal{M}^N$ and the functions $\Gamma^{\nu}_{\mu\lambda}(\mathbf{U})$ give a symmetric (Lévi-Cività) connection on $\mathcal{M}^N$ [11–13].

In the flat coordinates $n^1(\mathbf{U}), \ldots, n^N(\mathbf{U})$, the non-degenerate DN-bracket can be written in the constant form

$$\{n^{\nu}(X), n^{\mu}(Y)\} = e^{\nu} \delta^{\nu\mu} \delta'(X - Y)$$

where $e^{\nu} = \pm 1$.

The functionals

$$N^{\nu} = \int_{-\infty}^{+\infty} n^{\nu}(X) \, \mathrm{d}X$$

are the annihilators of the bracket (4.14) and the functional

$$P = \frac{1}{2} \int_{-\infty}^{+\infty} \sum_{\nu=1}^{N} e^{\nu} \, (n^{\nu}(X))^2 \, \mathrm{d}X$$

is the momentum functional generating the system $U^{\nu}_{T} = U^{\nu}_{X}$ according to (4.16).

The symplectic structure corresponding to non-degenerate DN-bracket has the weakly nonlocal form and can be written as

$$\Omega_{\nu\mu}(X, Y) = e^{\nu} \delta_{\nu\mu} \nu(X - Y)$$

in coordinates n^{ν} or, more generally,

$$\Omega_{\nu\mu}(X, Y) = \sum_{\lambda=1}^{N} e^{\lambda} \frac{\partial n^{\lambda}}{\partial U^{\nu}}(X) \nu(X - Y) \frac{\partial n^{\lambda}}{\partial U^{\mu}}(Y)$$

in arbitrary coordinates U^{ν}.

Let us also mention that the degenerate brackets (4.14) are more complicated but also have a good differential geometric structure [35].

Brackets (4.14) are closely connected with the integration theory of systems of hydrodynamic type (4.15). Namely, according to the conjecture of S P Novikov, all the diagonalizable systems (4.15) which are Hamiltonian with respect to DN-brackets (4.14) (with the Hamiltonian function of hydrodynamic type) are completely integrable. This conjecture was proved by Tsarev [64] who proposed a general procedure ('generalized Hodograph method') of integration of Hamiltonian diagonalizable systems (4.15).

In fact, Tsarev's 'generalized Hodograph method' permits us to integrate the wider class of diagonalizable systems (4.15) (semi-Hamiltonian systems, [64]) which appeared to be Hamiltonian, in more general (weakly nonlocal) Hamiltonian formalism.

The corresponding Poisson brackets (Mokhov–Ferapontov bracket and Ferapontov bracket) are the weakly nonlocal generalizations of DN-bracket (4.14) and are connected with geometry of submanifolds in pseudo-Euclidean spaces. Let us describe here the corresponding structures.

The Mokhov–Ferapontov bracket (MF-bracket) has the form [57]

$$\{U^{\nu}(X), U^{\mu}(Y)\} = g^{\nu\mu}(\mathbf{U})\delta'(X-Y) + b^{\nu\mu}_{\lambda}(\mathbf{U})U^{\lambda}_{X}\,\delta(X-Y) + c\,U^{\nu}_{X}\nu(X-Y)U^{\mu}_{Y} \tag{4.17}$$

As was proved in [57], expression (4.17) with $\det\|g^{\nu\mu}(\mathbf{U})\| \neq 0$ gives the Poisson bracket on the space $U^{\nu}(X)$ if and only if

(1) the tensor $g^{\nu\mu}(\mathbf{U})$ represents the pseudo-Riemannian contravariant metric of constant curvature c on the manifold $\mathcal{M}^N$, i.e.

$$R^{\nu\mu}_{\lambda\eta}(\mathbf{U}) = c(\delta^{\nu}_{\lambda}\delta^{\mu}_{\eta} - \delta^{\mu}_{\lambda}\delta^{\nu}_{\eta});$$

(2) the functions $\Gamma^{\nu}_{\mu\lambda}(\mathbf{U}) = -g_{\mu\alpha}(\mathbf{U})b^{\alpha\nu}_{\lambda}(\mathbf{U})$ represent the Lévi-Cività connection of metric $g_{\nu\mu}(\mathbf{U})$.

The Ferapontov bracket (F-bracket) is a more general weakly nonlocal generalization of the DN-bracket having the form [25–28]

$$\{U^{\nu}(X), U^{\mu}(Y)\} = g^{\nu\mu}(\mathbf{U})\delta'(X-Y) + b^{\nu\mu}_{\lambda}(\mathbf{U})U^{\lambda}_{X}\delta(X-Y) + \sum_{k=1}^{g} e_k w^{\nu}_{(k)\lambda}(\mathbf{U})U^{\lambda}_{X}\nu(X-Y)w^{\mu}_{(k)\delta}(\mathbf{U})U^{\delta}_{Y} \tag{4.18}$$

$e_k = \pm 1,\ \nu, \mu = 1, \ldots, N.$

Expression (4.18) (with $\det\|g^{\nu\mu}(\mathbf{U})\| \neq 0$) gives the Poisson bracket on the space $U^{\nu}(X)$ if and only if [25, 28]

(1) tensor $g^{\nu\mu}(\mathbf{U})$ represents the metric of the submanifold $\mathcal{M}^N \subset \mathbb{E}^{N+g}$ with flat normal connection in the pseudo-Euclidean space $\mathbb{E}^{N+g}$ of dimension $N+g$;

(2) the functions $\Gamma^{\nu}_{\mu\lambda}(\mathbf{U}) = -g_{\mu\alpha}(\mathbf{U})b^{\alpha\nu}_{\lambda}(\mathbf{U})$ represent the Lévi-Cività connection of metric $g_{\nu\mu}(\mathbf{U})$;

(3) the set of affinors $\{w^{\nu}_{(k)\lambda}(\mathbf{U})\}$ represent the full set of Weingarten operators corresponding to g linearly independent parallel vector fields in the normal bundle, such that

$$g_{\nu\tau}(\mathbf{U})w^{\tau}_{(k)\mu}(\mathbf{U}) = g_{\mu\tau}(\mathbf{U})w^{\tau}_{(k)\nu}(\mathbf{U}), \qquad \nabla_{\nu}w^{\mu}_{(k)\lambda}(\mathbf{U}) = \nabla_{\lambda}w^{\mu}_{(k)\nu}(\mathbf{U})$$

$$R^{\nu\mu}_{\lambda\eta}(\mathbf{U}) = \sum_{k=1}^{g} e_k\big(w^{\nu}_{(k)\lambda}(\mathbf{U})w^{\mu}_{(k)\eta}(\mathbf{U}) - w^{\mu}_{(k)\lambda}(\mathbf{U})w^{\nu}_{(k)\eta}(\mathbf{U})\big)$$

Besides that the set of affinors $w_{(k)}$ is commutative $[w_{(k)}, w_{(k')}] = 0$.

As was shown in [26] the expression (4.18) can be considered as the Dirac reduction of the Dubrovin–Novikov bracket connected with metric in $\mathbb{E}^{N+g}$ to the manifold $\mathcal{M}^N$ with a flat normal connection. Let us also note that the MF-bracket can be considered as a case of the F-bracket when $\mathcal{M}^N$ is a (pseudo)-sphere $\mathcal{S}^N \subset \mathbb{E}^{N+1}$ in a pseudo-Euclidean space.

The symplectic structures $\Omega_{\nu\mu}(X, Y)$ for both (non-degenerate) MF-bracket and F-bracket also have the weakly nonlocal form [53] and can be written in general coordinates U^{ν} as

$$\Omega_{\nu\mu}(X, Y) = \sum_{s=1}^{N+g} \epsilon_s \frac{\partial n^s}{\partial U^{\nu}}(X)\nu(X-Y)\frac{\partial n^s}{\partial U^{\mu}}(Y),$$

where $\epsilon_s = \pm 1$ and the metric G_{IJ} in the space $\mathbb{E}^{N+g}$ has the form $G_{IJ} = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_{N+g})$. The functions $n^1(\mathbf{U}), \ldots, n^{N+g}(\mathbf{U})$ are the ‘Canonical forms’ on the manifold $\mathcal{M}^N$ and play the role of densities and annihilators of bracket (4.18) and ‘Canonical Hamiltonian functions’ (see [53]) depending on the definition of phase space. In fact, the functions $n^s(\mathbf{U})$ are the restrictions of flat coordinates of metric G_{IJ} giving the DN-bracket in $\mathbb{E}^{N+g}$ on manifold $\mathcal{M}^N$. The mapping $\mathcal{M}^N \to \mathbb{E}^{N+g}$

$$(U^1, \ldots, U^N) \to (n^1(\mathbf{U}), \ldots, n^{N+g}(\mathbf{U}))$$

gives locally the embedding of $\mathcal{M}^N$ in $\mathbb{E}^{N+g}$ as a submanifold with a flat normal connection.

All the brackets (4.14), (4.17), (4.18) are connected with the Tsarev method of integration of systems (4.15). Namely, any diagonalizable system (4.15) Hamiltonian w.r.t. the (non-degenerate) bracket (4.14), (4.17) or (4.18) can be integrated by a 'generalized Hodograph method'.

We will not describe here the Tsarev method in detail. However, let us point out that the 'generalized Hodograph method' and the HT Hamiltonian structures were very useful for Whitham's systems obtained by the averaging of integrable PDEs [11, 12, 13, 29, 42, 43, 69].

The Hamiltonian approach to the Whitham method was started by Dubrovin and Novikov in [11] (see also [12, 13]) where the procedure of 'averaging' of a local field-theoretical Poisson bracket was proposed. The Dubrovin–Novikov procedure gives the DN-bracket for the Whitham system (4.15) in the case when the initial system is Hamiltonian w.r.t. a local Poisson bracket:

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k\geqslant o} B^{ij}_{(k)}(\varphi, \varphi_x, \ldots)\delta^{(k)}(x-y)$$

with the local Hamiltonian functional

$$H = \int_{-\infty}^{+\infty} h(\varphi, \varphi_x, \ldots)\,\mathrm{d}x.$$

The method of Dubrovin and Novikov is based on the presence of N (equal to the number of parameters U^ν of the family of m-phase solutions) local integrals

$$I^\nu = \int \mathcal{P}^\nu(\varphi, \varphi_x, \ldots)\,\mathrm{d}x \tag{4.19}$$

commuting with the Hamiltonian function and with each other

$$\{I^\nu, H\} = 0, \qquad \{I^\nu, I^\mu\} = 0 \tag{4.20}$$

and can be formulated in the following form.

We calculate the pairwise Poisson brackets of the densities $\mathcal{P}^\nu$ in the form

$$\{\mathcal{P}^\nu(x), \mathcal{P}^\mu(y)\} = \sum_{k\geqslant 0} A^{\nu\mu}_k(\varphi, \varphi_x, \ldots)\delta^{(k)}(x-y),$$

where

$$A^{\nu\mu}_0(\varphi, \varphi_x, \ldots) \equiv \partial_x Q^{\nu\mu}(\varphi, \varphi_x, \ldots)$$

according to (4.20). Then the Dubrovin–Novikov bracket on the space of functions $U(X)$ can be written in the form

$$\{U^\nu(X), U^\mu(Y)\} = \langle A^{\nu\mu}_1\rangle(U)\delta'(X-Y) + \frac{\partial\langle Q^{\nu\mu}\rangle}{\partial U^\gamma}U^\gamma_X\delta(X-Y), \tag{4.21}$$

where $\langle\cdots\rangle$ means the averaging on the family of m-phase solutions given by the formula

$$\langle F\rangle = \lim_{c\to\infty}\frac{1}{2c}\int_{-c}^{c} F(\varphi, \varphi_x, \ldots)\,\mathrm{d}x = \frac{1}{(2\pi)^m}\int_0^{2\pi}\ldots\int_0^{2\pi} F(\Phi, k^\alpha(U)\Phi_{\theta^\alpha}, \ldots)d^m\theta$$

and we choose the parameters U^ν such that they coincide with the values of I^ν on the corresponding solutions

$$U^\nu = \langle P^\nu(x)\rangle.$$

This procedure was generalized in [52] for the weakly nonlocal Hamiltonian structures. In this case, the procedure of construction of a general F-bracket (or an MF-bracket) for the Whitham system from the weakly non-local Poison bracket for the initial system was proposed[6]. In [54], the procedure of averaging of the weakly nonlocal symplectic structures was also suggested.

Here we will consider the construction of bracket (4.13) for the deformed Whitham system (3.16). Being considered for the KdV case, the procedure will have, in fact, general character and can be considered as a generalization of the Dubrovin–Novikov procedure for the deformed Whitham system in the general case.

We will use the Dirac restriction of a Poisson bracket on a submanifold to establish the procedure of the construction of a Poisson bracket of the form (4.13) for the deformed Whitham system (3.16), which we call the 'averaging' of a Poisson bracket for the deformed Whitham systems. Let us first introduce the Poisson brackets

$$\{\varphi(\theta, X), \varphi(\theta', Y)\} = \epsilon\delta(\theta - \theta')\delta'(X - Y) \tag{4.22}$$

and

$$\{\varphi(\theta, X), \varphi(\theta', Y)\} = \epsilon^3\delta(\theta - \theta')\delta'''(X - Y) + \epsilon\tfrac{2}{3}\varphi(X)\delta(\theta - \theta')\delta'(X - Y) + \epsilon\tfrac{1}{3}\varphi_X\delta(\theta - \theta')\delta(X - Y) \tag{4.23}$$

which correspond to the Gardner–Zakharov–Faddeev and the Magri brackets on the extended phase space $\varphi(\theta, X)$ periodic in θ with the period 2π. It is easy to see that both the expressions (4.22) and (4.23) give Poisson brackets on the extended functional space.

We now have to consider the 'subspace' in the extended functional space corresponding to the full family (3.17) parametrized by three functional parameters $(S(X), A(X), n(X))$.[7] We will now call the Dirac restriction of bracket (4.22) or (4.23) on the submanifold corresponding to the full family of solutions (3.17) the averaging of the Gardner–Zakharov–Faddeev bracket or the Magri bracket giving a Poisson bracket for the deformed Whitham system (3.16).

Let us recall that the Dirac restriction of a Poisson bracket on a submanifold $\mathcal{N}^k \subset \mathcal{M}^n$ is connected with the special choice of coordinates in the vicinity of the submanifold $\mathcal{N}^k$ which are divided to the 'coordinates on the submanifold' $(U^1, \ldots, U^k)$ and the constraints $(g^1, \ldots, g^{n-k})$ which define the submanifold $\mathcal{N}^k$. It is assumed that the submanifold $\mathcal{N}^k$ is given by the conditions

$$g^i(\mathbf{x}) = 0, \qquad i = 1, \ldots, n - k$$

while the k functions $U^1(\mathbf{x}), \ldots, U^k(\mathbf{x})$ on $\mathcal{M}^n$ play the role of the coordinate system on $\mathcal{N}^k$ after the restriction on this submanifold.

If the Hamiltonian flows generated by the functions $U^j(\mathbf{x})$ leave the submanifold $\mathcal{N}^k$ invariant, i.e. we have

$$\{U^j(\mathbf{x}), g^i(\mathbf{x})\} = 0 \qquad \text{for} \quad \mathbf{g}(\mathbf{x}) = 0$$

then the pairwise Poisson brackets of functions $U^j(\mathbf{x})$ give a Poisson tensor after the restriction on $\mathcal{N}^k$ with coordinates $(U^1, \ldots, U^k)$ which is called the Dirac restriction of the Poisson bracket $\{\ldots, \ldots\}$ defined on $\mathcal{M}^n$ on the submanifold $\mathcal{N}^k \subset \mathcal{M}^n$.

In general, according to Dirac procedure, if we have some constraints $g^i(\mathbf{x})$ which define a submanifold $\mathcal{N}^k$ and some functions $U^j(\mathbf{x})$ giving a coordinate system on $\mathcal{N}^k$ we have to find k linear combinations $\beta^j_s(\mathbf{U})g^s(\mathbf{x})$ at every point of $\mathcal{N}^k$ such that we have for the functions

$$\tilde{U}^j(\mathbf{x}) = U^j(\mathbf{x}) + \beta^j_s(\mathbf{U})g^s(\mathbf{x}), \qquad j = 1, \ldots, k$$

[6] The final proof of the Jacobi identity for the bracket given by the Dubrovin–Novikov procedure was given in [51].

[7] Let us note that all the constructions are considered here just on the level of the formal asymptotic series.

the relations

$$\{\tilde{U}^j(\mathbf{x}), g^i(\mathbf{x})\} = 0 \qquad \text{at} \quad \mathbf{g}(\mathbf{x}) = 0.$$

The functions $\tilde{U}^j(\mathbf{x})$ have the same values as the functions $U^j(\mathbf{x})$ at the points of $\mathcal{N}^k$ and we can then define the Dirac bracket $\{\ldots,\ldots\}_D$ on $\mathcal{N}^k$ by the formula

$$\{U^i, U^j\}_D = \{\tilde{U}^i(\mathbf{x}), \tilde{U}^j(\mathbf{x})\}|_{\mathcal{N}^k}(\mathbf{U}).$$

The functions $\beta^j_s(\mathbf{U})$ are defined from the linear system

$$\{g^i(\mathbf{x}), g^s(\mathbf{x})\}|_{\mathcal{N}^k}\beta^j_s(\mathbf{U}) + \{g^i(\mathbf{x}), U^j(\mathbf{x})\}|_{\mathcal{N}^k} = 0, \qquad i = 1,\ldots,n-k$$

and we can also write

$$\{U^i, U^j\}_D = \{U^i(\mathbf{x}), U^j(\mathbf{x})\}|_{\mathcal{N}^k} - \beta^i_s(\mathbf{U})\{g^s(\mathbf{x}), g^q(\mathbf{x})\}|_{\mathcal{N}^k}\beta^j_q(\mathbf{U})$$

for the Dirac bracket on $\mathcal{N}^k$.

Let us now describe the Dirac procedure in our situation.

First, let us introduce new coordinates on the submanifold $\mathcal{K}$ corresponding to solutions (3.17) based on the conservation laws of the KdV equation (3.18).

Let us choose three integrals of the KdV equation such that their values on the family of one-phase solutions of KdV are functionally independent. In our case it is most convenient to take the integrals

$$I_0 = \int_{-\infty}^{+\infty} \varphi\, \mathrm{d}x, \qquad I_1 = \int_{-\infty}^{+\infty} \frac{\varphi^2}{2}\, \mathrm{d}x, \qquad I_2 = \int_{-\infty}^{+\infty} \left(\frac{\varphi^2}{6} - \frac{\varphi_x^2}{2}\right) \mathrm{d}x.$$

The integrals transform naturally to the integrals of the KdV equation (3.18) on the extended phase space

$$I_0 = \int_{-\infty}^{+\infty}\int_0^{2\pi} \varphi\, \mathrm{d}X \frac{\mathrm{d}\theta}{2\pi}, \qquad I_1 = \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{\varphi^2}{2}\, \mathrm{d}X \frac{\mathrm{d}\theta}{2\pi}$$
$$I_2 = \int_{-\infty}^{+\infty}\int_0^{2\pi} \left(\frac{\varphi^2}{6} - \frac{\epsilon^2}{2}\varphi_X^2\right) \mathrm{d}X \frac{\mathrm{d}\theta}{2\pi}.$$

Let us now introduce the functionals

$$U^\nu(X) = \int_0^{2\pi} \mathcal{P}^\nu(\varphi, \epsilon\varphi_X, \ldots) \frac{\mathrm{d}\theta}{2\pi}, \qquad \nu = 0, 1, \ldots,$$

i.e.

$$\begin{aligned} U^0(X) &= \int_0^{2\pi} \varphi(\theta, X) \frac{\mathrm{d}\theta}{2\pi}, \\ U^1(X) &= \frac{1}{2}\int_0^{2\pi} \varphi^2(\theta, X) \frac{\mathrm{d}\theta}{2\pi} \\ U^2(X) &= \int_0^{2\pi} \left(\frac{\varphi^2}{6}(\theta, X) - \frac{\epsilon^2}{2}\varphi_X^2(\theta, X)\right) \frac{\mathrm{d}\theta}{2\pi} \end{aligned} \tag{4.24}$$

and consider the values of the functionals $U^\nu(X)$ on the submanifold $\mathcal{K}$.

It is easy to see that the values of $U^\nu(X)$ on $\mathcal{K}$ are equal in the main approximation to the values of the functionals I^ν on the one-phase solutions of KdV with the parameters $(k(X), A(X), n(X))$ and have in general higher corrections polynomial in derivatives of the functions $k(X)$, $A(X)$, and $n(X)$. It is also not difficult to see that the higher corrections to $U^\nu(X)$ contain only even degrees in the expansion w.r.t. the derivatives k_X, A_X, $n_X, \ldots$

for our choice of the initial phase of the functions $\Phi(\theta, k, A, n)$ view the statements of the lemma 3.1. Thus for the functionals $U^0(X)$, $U^1(X)$, $U^2(X)$, we can write

$$\begin{aligned} U^0(X) &= \langle\varphi\rangle_0(X) \equiv n(X) \\ U^1(X) &= \langle\varphi^2\rangle_0(X) + \sum_{s\geqslant 1} U^1_{2s} \\ U^2(X) &= \left\langle \frac{\varphi^2}{6} - \frac{\varphi_x^2}{2} \right\rangle_0 (X) + \sum_{s\geqslant 1} U^2_{2s}, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} \langle\varphi\rangle_0 &\equiv \int_0^{2\pi} \Phi(\theta, k, A, n) \frac{\mathrm{d}\theta}{2\pi} \equiv n \\ \langle\varphi^2\rangle_0 &\equiv \int_0^{2\pi} \Phi^2(\theta, k, A, n) \frac{\mathrm{d}\theta}{2\pi} \\ \left\langle \frac{\varphi^2}{6} - \frac{\varphi_x^2}{2} \right\rangle_0 &= \int_0^{2\pi} \left(\frac{1}{6}\Phi^3(\theta, k, A, n) - \frac{k^2}{2}\Phi_\theta^2(\theta, k, A, n) \right) \frac{\mathrm{d}\theta}{2\pi} \end{aligned}$$

and the values U^1_{2s}, U^2_{2s} are graded polynomials in the derivatives of k, A and n having degree $2s$.

Since the values of I^0, I^1, I^2 are functionally independent on the space of one-phase solutions of KdV, we can write the 'inverted series' for the functions $k(X)$, $A(X)$, $n(X)$. We have to change the gradation rules now such that we will define the gradation degree with respect to the X-derivatives of the parameters (U^0, U^1, U^2) instead of (k, A, n). So we can now write

$$\begin{aligned} k(X) &= k_0(U^0(X), U^1(X), U^2(X)) + \sum_{s\geqslant 1} k_{(2s)}([U^0, U^1, U^2], X) \\ A(X) &= A_0(U^0(X), U^1(X), U^2(X)) + \sum_{s\geqslant 1} A_{(2s)}([U^0, U^1, U^2], X) \\ n(X) &= U^0(X), \end{aligned} \tag{4.26}$$

where $k_0(U^0, U^1, U^2)$, $A_0(U^0, U^1, U^2)$ are the exact 'one-phase' expressions for the parameters k and A in terms of (U^0, U^1, U^2) and the functions $k_{(2s)}$, $A_{(2s)}$ are graded polynomials in the derivatives of (U^0, U^1, U^2) having degree $2s$.

Using relations (4.26) we can re-expand also solutions (3.17) as graded series with respect to the X-derivatives of the values of the functionals $U^0(X)$, $U^1(X)$, $U^2(X)$ on $\mathcal{K}$ every time, such that we have

$$\phi(\theta, X, T) = \Phi^{\mathbf{U}}\left(\frac{S(X,T)}{\epsilon} + \theta, U^0, U^1, U^2\right) + \sum_{l\geqslant 1} \epsilon^l \Phi^{\mathbf{U}}_{(l)}\left(\frac{S(X,T)}{\epsilon} + \theta, X, T\right), \tag{4.27}$$

where all $\Phi^{\mathbf{U}}_{(l)}$ are the graded polynomials of $\left(U^0_X, U^1_X, U^2_X, \ldots\right)$ of degree l.

The function $\Phi^{\mathbf{U}}(\theta, U^0, U^1, U^2)$ represents the exact one-phase solution of KdV depending on the parameters (U^0, U^1, U^2) and we have by definition

$$\Phi^{\mathbf{U}}(\theta, U^0, U^1, U^2) = \Phi(\theta, k(\mathbf{U}), A(\mathbf{U}), n(\mathbf{U})).$$

According to our approach we will assume that series (4.27) and (3.17) are equivalent representations of formal asymptotic solutions (3.17) connected by the change of the asymptotic functional parameters (4.25) and (4.26). Let us also note that due to the form of relations (4.25) and (4.26) the symmetric properties of the functions $\Phi^{\mathbf{U}}_{(l)}(\theta, X, T)$ remain the same as those for the terms of series (3.17), i.e. we have

$$\Phi^{\mathbf{U}}_{(2s)}(-\theta, X, T) = \Phi^{\mathbf{U}}_{(2s)}(\theta, X, T), \qquad \Phi^{\mathbf{U}}_{(2s+1)}(-\theta, X, T) = -\Phi^{\mathbf{U}}_{(2s+1)}(\theta, X, T), \quad s \geqslant 0.$$

We can assume in the same way that the functions of the submanifold $\mathcal{K}$ are represented now by the asymptotic series

$$\varphi(\theta, X) = \Phi^{\mathbf{U}}\left(\frac{S(X)}{\epsilon} + \theta, U^0, U^1, U^2\right) + \sum_{l\geqslant 1} \epsilon^l \Phi^{\mathbf{U}}_{(l)}\left(\frac{S(X)}{\epsilon} + \theta, [U^0, U^1, U^2], X\right),$$

so the functionals $U^0(X), U^1(X), U^2(X)$ play the role of coordinates on this submanifold and these are exactly the functionals we are going to use for the Dirac procedure.

Let us introduce now the system of 'constraints' which defines our submanifold $\mathcal{K}$ in the functional space. For our purposes it will be convenient to write the constraints in the following form.

Let us denote

$$\psi(\theta, [\mathbf{U}], X) = \Phi^{\mathbf{U}}(\theta, \mathbf{U}, X) + \sum_{l\geqslant 1} \epsilon^l \Phi^{\mathbf{U}}_{(l)}(\theta, [\mathbf{U}], X),$$

where the notations $\mathbf{U}(\mathbf{X}) = (U^0(X), U^1(X), U^2(X))$ denote now the functionals (4.24) defined on the full functional space $\{\varphi(\theta, X)\}$. We introduce now the constraints $g(\theta, X)$ by the formula

$$g(\theta, X) = \varphi(\theta, X) - \psi\left(\frac{S[\mathbf{U}](X)}{\epsilon} + \theta, [\mathbf{U}], X\right) \tag{4.28}$$

as the functionals on the space $\mathcal{M}$. It is evident that the relations

$$g(\theta, X) = 0$$

define then exactly the 'sub-manifold' $\mathcal{K}$ we consider here.

However, the set of constraints (4.28) is certainly not independent in the ordinary sense. Namely, in the full analogy with the finite-dimensional case the following relations take place identically for the 'gradients' $\delta g(\theta, X)/\delta\varphi(\theta', Y)$ on the 'sub-manifold' $\mathcal{K}$:

$$\int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{\delta U^\nu(Z)}{\delta\varphi(\theta, X)} \frac{\delta g(\theta, X)}{\varphi(\theta', Y)} \frac{d\theta}{2\pi}\, dX \equiv 0. \tag{4.29}$$

Nevertheless, it will be convenient for us not to choose an independent system of constraints and to keep constraints (4.28) for our purposes, so we have to remember the presence of relations (4.29) for system (4.28).

For the Dirac restriction of bracket (4.22) or (4.23) on the submanifold $\mathcal{K}$, we have to modify now the functionals $U^0(X), U^1(X), U^2(X)$ by the linear combinations of constraints $g(\theta, X)$:

$$\tilde{U}^\nu(X) = U^\nu(X) + \int_{-\infty}^{+\infty}\int_0^{2\pi} g(\theta, Y)\beta^\nu\left(\frac{S[\mathbf{U}](Y)}{\epsilon} + \theta, [\mathbf{U}], Y, X\right)\frac{d\theta}{2\pi}\, dY$$

such that the functionals $\tilde{U}^\nu(X)$ leave invariant the submanifold $\mathcal{K}$ in the corresponding Hamiltonian structure and then to use the functionals $\tilde{U}^\nu(X)$ for the construction of the Dirac bracket on $\mathcal{K}$. The functions $\beta^\nu(S(Y)/\epsilon + \theta, Y, X)$ should satisfy the relation

$$\int_{-\infty}^{+\infty}\int_0^{2\pi} \{g(\theta, X), g(\theta', Z)\}\beta^\nu\left(\frac{S(Z)}{\epsilon} + \theta', Z, Y\right)\frac{d\theta'}{2\pi}\, dZ + \{g(\theta, X), U^\nu(Y)\} = 0 \tag{4.30}$$

on $\mathcal{K}$ and are defined at every 'point' of $\mathcal{K}$ modulo the linear combinations of the functions $\delta U^\mu(W)/\delta\varphi(\theta, Y)$ view the original dependence of constraints (4.28).

The Dirac bracket on the manifold $\mathcal{K}$ can be defined by the formula

$$\{U^{\nu}(X), U^{\mu}(Y)\}_D = \{U^{\nu}(X), U^{\mu}(Y)\}|_{\mathcal{K}} - \int \beta^{\nu}\left(\frac{S(X)}{\epsilon} + \theta, Z, X\right) \times \{g(\theta, Z), g(\theta', W)\}|_{\mathcal{K}} \beta^{\mu}\left(\frac{S(Y)}{\epsilon} + \theta', W, Y\right) \frac{d\theta}{2\pi}\frac{d\theta'}{2\pi}\, dZ\, dW \qquad (4.31)$$

so the procedure gives a unique definition of the bracket $\{U^{\nu}(X), U^{\mu}(Y)\}_D$.

To obtain a local deformed Poisson bracket on $\mathcal{K}$ we will try to find the functions $\beta^{\nu}(\theta, Y, X)$ in the form

$$\beta^{\nu}(\theta, Y, X) = \sum_{s\geqslant 1} \epsilon^s \beta^{\nu}_{(s)}(\theta, Y, X), \qquad (4.32)$$

where the functions $\beta^{\nu}_{(s)}(\theta, Y, X)$ are represented as the local distributions

$$\beta^{\nu}_{(s)}(\theta, Y, X) = \sum_{p=0}^{s} \beta^{\nu}_{(s),p}(\theta, Y)\delta^{(p)}(Y - X) \qquad (4.33)$$

having gradation s assuming that the derivatives of the delta-function $\delta^{(p)}(Y - X)$ have degree p by definition.

Thus, we assume that all the functions $\beta^{\nu}_{(s),p}(\theta, Y)$ on $\mathcal{K}$ are local functionals of $\big(U^0(X), U^1(X), U^2(X), U^0_X, U^1_X, U^2_X, \dots\big)$ at every θ, polynomial in derivatives $\big(U^0_X, U^1_X, U^2_X, \dots\big)$ and having degree $s-p$ according to our previous definition. This structure of $\beta^{\nu}(\theta, Y, X)$ is obviously equivalent to the statement that the functionals

$$\int_{-\infty}^{+\infty} U^{\nu}(X) q(X)\, dX$$

with a 'slow' function of X $q(X)$ can be modified with the aid of a linear combination of constraints (4.28) with the coefficients

$$B^{\nu}_{[q]}(\theta, Y) = \sum_{s\geqslant 1} \epsilon^s \sum_{p=0}^{s} \beta^{\nu}_{(s),p}(\theta, Y) \frac{d^p q(X)}{dX^p}$$

to leave the submanifold $\mathcal{K}$ invariant. According to this scheme, the derivatives $d^s q/dX^s$ of the slow function $q(X)$ have degree s as well as the derivatives of the parameters $U^0(X)$, $U^1(X)$, $U^2(X)$.

Finally, we have to study now system (4.30) for the cases of the Gardner–Zakharov–Faddeev bracket and the Magri bracket to investigate the possibility of finding the functions $\beta^{\nu}(\theta, Y, X)$ in the form (4.32)–(4.33). Let us formulate here the following theorem.

Theorem 4.2. *For both the Gardner–Zakharov–Faddeev bracket and the Magri bracket for KdV, the functions $\beta^{\nu}(\theta, Y, X)$ can be found in the form (4.32)–(4.33) on the family $\mathcal{K}$. Thus, the Dirac restriction of both the brackets on the family $\mathcal{K}$ has the local deformed Hydrodynamic form (4.13) which gives two deformed hydrodynamic type brackets for the deformed Whitham system (3.16).*

Proof. Let us analyze equations (4.30) for the case of the Gardner–Zakharov–Faddeev bracket and the Magri bracket. We first have on the family $\mathcal{K}$

$$\{g(\theta, X), g(\theta', Z)\}|_{\mathcal{K}} = \{\varphi(\theta, X), \varphi(\theta', Z)\}|_{\mathcal{K}}$$
$$- \int_{-\infty}^{+\infty} \mathrm{d}W \{\varphi(\theta, X), U^{\mu}(W)\}|_{\mathcal{K}} \frac{\delta\psi(S[\mathbf{U}](Z)/\epsilon + \theta', [\mathbf{U}], Z)}{\delta U^{\mu}(W)}$$
$$- \int_{-\infty}^{+\infty} \mathrm{d}W \frac{\delta\psi(S[\mathbf{U}](X)/\epsilon + \theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)} \{U^{\mu}(W), \varphi(\theta', Z)\}|_{\mathcal{K}}$$
$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}W\, \mathrm{d}V \frac{\delta\psi(S[\mathbf{U}](X)/\epsilon + \theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)}$$
$$\times \{U^{\mu}(W), U^{\gamma}(V)\}|_{\mathcal{K}} \frac{\delta\psi(S[\mathbf{U}](Z)/\epsilon + \theta', [\mathbf{U}], Z)}{\delta U^{\gamma}(V)}$$

(summation over repeated indices)[8].

In the same way,

$$\{g(\theta, X), U^{\nu}(Y)\}|_{\mathcal{K}} = \{\varphi(\theta, X), U^{\nu}(Y)\}|_{\mathcal{K}}$$
$$- \int_{-\infty}^{+\infty} \mathrm{d}W \frac{\delta\psi(S[\mathbf{U}](X)/\epsilon + \theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)} \{U^{\mu}(W), U^{\nu}(Y)\}|_{\mathcal{K}.}$$

Let us now mention the Poisson bracket $\{k(W), U^{\nu}(Y)\}|_{\mathcal{K}}$. As we saw already the functional

$$I^{\nu} = \int_{-\infty}^{+\infty} U^{\nu}(Y)\, \mathrm{d}Y$$

leaves invariant the submanifold $\mathcal{K}$ so the Poisson bracket $\{k(W), I^{\nu}\}|_{\mathcal{K}}$ should give exactly the Whitham evolution of the functional $k([\mathbf{U}], W)$ corresponding to the ν-flow of the KdV hierarchy. So we have

$$\{k(W), I^{\nu}\}|_{\mathcal{K}} = \epsilon \left(\omega^{\nu}(k, A, n) + \sum_{l \geqslant 1} \epsilon^{2l} \omega^{\nu}_{(2l)}(k, A, n, \ldots) \right)_W$$
$$= \epsilon \left(\omega^{\nu}_0(\mathbf{U}) + \sum_{l \geqslant 1} \epsilon^{2l} \tilde{\omega}^{\nu}_{(2l)}(\mathbf{U}, \mathbf{U}_X, \ldots) \right)_W$$

with some functionals $\tilde{\omega}^{\nu}_{(2l)}(\mathbf{U}, \mathbf{U}_X, \ldots)$ according to (4.5).

According to the structure of the bracket $\{k(W), U^{\nu}(Y)\}|_{\mathcal{K}}$, we should have then

$$\{k(W), U^{\nu}(Y)\}|_{\mathcal{K}} = \epsilon \left(\omega^{\nu}_0(\mathbf{U}(W)) + \sum_{l \geqslant 1} \epsilon^{2l} \tilde{\omega}^{\nu}_{(2l)}(\mathbf{U}, \mathbf{U}_W, \ldots) \right)_W \delta(W - Y)$$
$$+ \sum_{s \geqslant 1} \epsilon^{s} \kappa^{\nu L}_{(s)}(\mathbf{U}, \mathbf{U}_W, \ldots) \delta^{(s)}(W - Y),$$

where $\kappa^{\nu L}_{(s)}$ are some local functionals of $(\mathbf{U}, \mathbf{U}_W, \ldots)$ given by sums of terms of degree $\geqslant 0$.

[8] Let us note that we assume the differentiation $\delta/\delta U$ in the sense of the values of functionals $U^{\nu}(X)$ on the family $\mathcal{K}$ while we treat $U^{\nu}(X)$ inside the brackets as a functional on the whole functional space.

We can write then

$$\frac{1}{2\epsilon}\int_{-\infty}^{+\infty} \mathrm{d}W\,\psi_\theta(\theta,[\mathbf{U}],X)\,\mathrm{sgn}(X-W)\{k(W),U^\nu(Y)\}|_{\mathcal{K}}$$

$$= \psi_\theta(\theta,[\mathbf{U}],X)\left(\omega_0^\nu(\mathbf{U}(X)) + \sum_{l\geqslant 1}\epsilon^{2l}\tilde{\omega}^\nu_{(2l)}(\mathbf{U},\mathbf{U}_X,\ldots)\right)\delta(X-Y)$$

$$-\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}W\,\psi_\theta(\theta,[\mathbf{U}],X)\,\mathrm{sgn}(X-W)$$

$$\times\left(\omega_0^\nu(\mathbf{U}(W)) + \sum_{l\geqslant 1}\epsilon^{2l}\tilde{\omega}^\nu_{(2l)}(\mathbf{U},\mathbf{U}_W,\ldots)\right)\delta'(W-Y)$$

$$+\frac{1}{2\epsilon}\int_{-\infty}^{+\infty}\mathrm{d}W\,\psi_\theta(\theta,[\mathbf{U}],X)\,\mathrm{sgn}(X-W)\left(\sum_{s\geqslant 1}\epsilon^s\kappa^{\nu L}_{(s)}(\mathbf{U},\mathbf{U}_W,\ldots)\delta^{(s)}(W-Y)\right).$$

In the same way, we obtain

$$\{\varphi(\theta,X),I^\nu\}|_{\mathcal{K}} = \psi_\theta\left(\frac{S(X)}{\epsilon}+\theta,[\mathbf{U}],X\right)\left(\omega_0^\nu(\mathbf{U}(X)) + \sum_{l\geqslant 1}\epsilon^{2l}\tilde{\omega}^\nu_{(2l)}(\mathbf{U},\mathbf{U}_X,\ldots)\right)$$

$$+\int_{-\infty}^{+\infty}\mathrm{d}W\,\psi_{U^\mu}\left(\frac{S(X)}{\epsilon}+\theta,[\mathbf{U}],X,W\right)\{U^\mu(W),I^\nu\}|_{\mathcal{K}},$$

where

$$\psi_{U^\mu}(\theta,[\mathbf{U}],X,W) \equiv \frac{\delta\psi(\theta,[\mathbf{U}],X)}{\delta U^\mu(W)}.$$

So, from the structure of the bracket $\{\varphi(\theta,X),U^\nu(Y)\}|_{\mathcal{K}}$, we can conclude

$$\chi_L^\nu\left(\frac{S(X)}{\epsilon}+\theta,X,Y\right) \equiv \{\varphi(\theta,X),U^\nu(Y)\}|_{\mathcal{K}}$$

$$= \psi_\theta\left(\frac{S(X)}{\epsilon}+\theta,[\mathbf{U}],X\right)\left(\omega_0^\nu(\mathbf{U}(X)) + \sum_{l\geqslant 1}\epsilon^{2l}\tilde{\omega}^\nu_{(2l)}(\mathbf{U},\mathbf{U}_X,\ldots)\right)\delta(X-Y)$$

$$+\int_{-\infty}^{+\infty}\mathrm{d}W\,\psi_{U^\mu}\left(\frac{S(X)}{\epsilon}+\theta,[\mathbf{U}],X,W\right)\{U^\mu(W),U^\nu(Y)\}|_{\mathcal{K}}$$

$$+\sum_{s\geqslant 1}\epsilon^s\lambda^{\nu L}_{(s)}\left(\frac{S(X)}{\epsilon}+\theta,[\mathbf{U}],X\right)\delta^{(s)}(X-Y) \tag{4.34}$$

for some local functionals $\lambda^{\nu L}_{(s)}(\theta,[\mathbf{U}],X)$ on $\mathcal{K}$, polynomial in the derivatives $(\mathbf{U}_X,\mathbf{U}_{XX},\ldots)$ and given by sums of terms of degree $\geqslant 0$.

In the same way, we put

$$\chi_R^\nu\left(Y,X,\frac{S(X)}{\epsilon}+\theta,\right) \equiv \{U^\nu(Y),\varphi(\theta,X)\}|_{\mathcal{K}}$$

$$= -\delta(Y-X)\left(\omega_0^\nu(\mathbf{U}(X)) + \sum_{l\geqslant 1}\epsilon^{2l}\tilde{\omega}^\nu_{(2l)}(\mathbf{U},\mathbf{U}_X,\ldots)\right)\psi_\theta\left(\frac{S(X)}{\epsilon}+\theta,[\mathbf{U}],X\right)$$

$$+\int_{-\infty}^{+\infty} \mathrm{d}W\, \{U^{\nu}(Y), U^{\mu}(W)\}|_{\mathcal{K}}\, \psi_{U^{\mu}} \left(\frac{S(X)}{\epsilon} + \theta, [\mathbf{U}], X, W\right)$$
$$+\sum_{s\geqslant 1} \epsilon^{s} \delta^{(s)}(Y - X) \lambda^{\nu R}_{(s)} \left(\frac{S(X)}{\epsilon} + \theta, [\mathbf{U}], X\right). \tag{4.35}$$

Let us also denote

$$\zeta_{L}\left(\frac{S(X)}{\epsilon} + \theta, X, Y,\right) \equiv \{\varphi(\theta, X), k(Y)\}|_{\mathcal{K}}$$
$$\zeta_{R}\left(Y, X, \frac{S(X)}{\epsilon} + \theta,\right) \equiv \{k(Y), \varphi(\theta, X)\}|_{\mathcal{K}}.$$

We have now

$$\alpha^{\nu}\left(\frac{S(X)}{\epsilon} + \theta, X, Y\right) \equiv \{g(\theta, X), U^{\nu}(Y)\}|_{\mathcal{K}}$$
$$= \{\varphi(\theta, X), U^{\nu}(Y)\}|_{\mathcal{K}} - \int_{-\infty}^{+\infty} \mathrm{d}W \psi_{U^{\mu}} \left(\frac{S(X)}{\epsilon} + \theta, [\mathbf{U}], X, W\right) \{U^{\mu}(W), U^{\nu}(Y)\}|_{\mathcal{K}}$$
$$- \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} \mathrm{d}W \psi_{\theta} \left(\frac{S(X)}{\epsilon} + \theta, [\mathbf{U}], X\right) \operatorname{sgn}(X - W) \{k(W), U^{\nu}(Y)\}|_{\mathcal{K}}$$
$$= \sum_{s\geqslant 1} \epsilon^{s} \lambda^{\nu L}_{(s)} \left(\frac{S(X)}{\epsilon} + \theta, [\mathbf{U}], X\right) \delta^{(s)}(X - Y)$$
$$+ \frac{1}{2} \int_{-\infty}^{+\infty} \mathrm{d}W \psi_{\theta} \left(\frac{S(X)}{\epsilon} + \theta, [\mathbf{U}], X\right) \operatorname{sgn}(X - W)$$
$$\times \left(\omega^{\nu}_{0}(W) + \sum_{l\geqslant 1} \epsilon^{2l} \tilde{\omega}^{\nu}_{(2l)}(W)\right) \delta'(W - Y)$$
$$- \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} \mathrm{d}W \psi_{\theta} \left(\frac{S(X)}{\epsilon} + \theta, [\mathbf{U}], X\right) \operatorname{sgn}(X - W) \left(\sum_{s\geqslant 1} \epsilon^{s} \kappa^{\nu L}_{(s)}(W) \delta^{(s)}(W - Y)\right).$$

Using the same arguments, we obtain that for the case of the Gardner–Zakharov–Faddeev bracket we have the following equation for the functions $\beta^{\nu}(\theta, Z, Y)$:

$$\int_{-\infty}^{+\infty} \int_{0}^{2\pi} L(\theta, \theta', X, Z) \beta^{\nu}(\theta', Z, Y) \frac{\mathrm{d}\theta'}{2\pi}\, \mathrm{d}Z = \alpha^{\nu}(\theta, X, Y), \tag{4.36}$$

where

$$L(\theta, \theta', X, Z) = k\delta'(\theta - \theta')\delta(X - Z) + \epsilon\delta(\theta - \theta')\delta'(X - Z)$$
$$- \int_{-\infty}^{+\infty} \mathrm{d}W \chi^{\mu}_{L}(\theta, X, W) \frac{\delta\psi(\theta', [\mathbf{U}], Z)}{\delta U^{\mu}(W)} - \int_{-\infty}^{+\infty} \mathrm{d}W \frac{\delta\psi(\theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)} \chi^{\mu}_{R}(W, Z, \theta')$$
$$+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}W\, \mathrm{d}V \frac{\delta\psi(\theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)} \{U^{\mu}(W), U^{\gamma}(V)\}|_{\mathcal{K}} \frac{\delta\psi(\theta', [\mathbf{U}], Z)}{\delta U^{\gamma}(V)}$$
$$- \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} \mathrm{d}W \zeta_{L}(\theta, X, W) \operatorname{sgn}(Z - W) \psi_{\theta'}(\theta', [\mathbf{U}], Z)$$
$$- \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} \mathrm{d}W \psi_{\theta}(\theta, [\mathbf{U}], X) \operatorname{sgn}(X - W) \zeta_{R}(W, Z, \theta')$$

$$+ \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}W \, \mathrm{d}V \frac{\delta \psi(\theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)} \{U^{\mu}(W), k(V)\}|_{\mathcal{K}} \mathrm{sgn}(Z - V) \psi_{\theta'}(\theta', [\mathbf{U}], Z)$$
$$+ \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}W \, \mathrm{d}V \psi_{\theta}(\theta, [\mathbf{U}], X) \, \mathrm{sgn}(X - W) \{k(W), U^{\gamma}(V)\}|_{\mathcal{K}} \frac{\delta \psi(\theta', [\mathbf{U}], Z)}{\delta U^{\gamma}(V)}$$
$$+ \frac{1}{4\epsilon^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}W \, \mathrm{d}V \psi_{\theta}(\theta, [\mathbf{U}], X) \, \mathrm{sgn}(X - W) \{k(W), k(V)\}|_{\mathcal{K}}$$
$$\times \, \mathrm{sgn}(Z - V) \psi_{\theta'}(\theta', [\mathbf{U}], Z).$$

Let us note now that the bracket $\{U^{\nu}(X), U^{\mu}(Y)\}|_{\mathcal{K}}$ has the order $\mathcal{O}(\epsilon)$ and, besides that, its main term in the ϵ-expansion coincides precisely with the Dubrovin–Novikov bracket defined above.

Let us put now the additional condition

$$\int_{0}^{2\pi} \psi_{\theta}(\theta, [\mathbf{U}], Z) \beta^{\nu}(\theta, Z, Y) \frac{\mathrm{d}\theta}{2\pi} \equiv 0 \tag{4.37}$$

which will be confirmed *a posteriori* for our $\beta^{\nu}(\theta, Z, Y)$. We can reduce then the operator $L(\theta, \theta', X, Z)$ to the form

$$L_{\mathrm{eff}}(\theta, \theta', X, Z) = k\delta'(\theta - \theta')\delta(X - Z) + \epsilon \delta(\theta - \theta')\delta'(X - Z)$$
$$- \int_{-\infty}^{+\infty} \mathrm{d}W \chi_{L}^{\mu}(\theta, X, W) \frac{\delta \psi(\theta', [\mathbf{U}], Z)}{\delta U^{\mu}(W)} - \int_{-\infty}^{+\infty} \mathrm{d}W \frac{\delta \psi(\theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)} \chi_{R}^{\mu}(W, Z, \theta')$$
$$+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}W \, \mathrm{d}V \frac{\delta \psi(\theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)} \{U^{\mu}(W), U^{\gamma}(V)\}|_{\mathcal{K}} \frac{\delta \psi(\theta', [\mathbf{U}], Z)}{\delta U^{\gamma}(V)}$$
$$- \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} \mathrm{d}W \psi_{\theta}(\theta, [\mathbf{U}], X) \, \mathrm{sgn}(X - W) \zeta_{R}(W, Z, \theta')$$
$$+ \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}W \, \mathrm{d}V \psi_{\theta}(\theta, [\mathbf{U}], X) \, \mathrm{sgn}(X - W) \{k(W), U^{\gamma}(V)\}|_{\mathcal{K}} \frac{\delta \psi(\theta', [\mathbf{U}], Z)}{\delta U^{\gamma}(V)}.$$

Let us define now the functions $\beta^{\nu}(\theta, Z, Y)$ as the solutions of the equations

$$\int_{-\infty}^{+\infty} \int_{0}^{2\pi} L_{\mathrm{eff}}^{I}(\theta, \theta', X, Z) \beta^{\nu}(\theta', Z, Y) \frac{\mathrm{d}\theta'}{2\pi} \, \mathrm{d}Z = \alpha^{\nu I}(\theta, X, Y), \tag{4.38}$$

where

$$L_{\mathrm{eff}}(\theta, \theta', X, Z) = L_{\mathrm{eff}}^{I}(\theta, \theta', X, Z) + L_{\mathrm{eff}}^{II}(\theta, \theta', X, Z)$$
$$\alpha^{\nu}(\theta, X, Y) = \alpha^{\nu I}(\theta, X, Y) + \alpha^{\nu II}(\theta, X, Y)$$

and

$$L_{\mathrm{eff}}^{I}(\theta, \theta', X, Z) = k\delta'(\theta - \theta')\delta(X - Z) + \epsilon \delta(\theta - \theta')\delta'(X - Z)$$
$$- \int_{-\infty}^{+\infty} \mathrm{d}W \chi_{L}^{\mu}(\theta, X, W) \frac{\delta \psi(\theta', [\mathbf{U}], Z)}{\delta U^{\mu}(W)} - \int_{-\infty}^{+\infty} \mathrm{d}W \frac{\delta \psi(\theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)} \chi_{R}^{\mu}(W, Z, \theta')$$
$$+ \psi_{\theta}(\theta, [\mathbf{U}], X) \left(\omega_{0}^{\mu}(X) + \sum_{l \geqslant 1} \epsilon^{2l} \tilde{\omega}_{(2l)}^{\mu}(X) \right) \frac{\delta \psi(\theta', [\mathbf{U}], Z)}{\delta U^{\mu}(X)}$$
$$+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}W \, \mathrm{d}V \frac{\delta \psi(\theta, [\mathbf{U}], X)}{\delta U^{\mu}(W)} \{U^{\mu}(W), U^{\gamma}(V)\}|_{\mathcal{K}} \frac{\delta \psi(\theta', [\mathbf{U}], Z)}{\delta U^{\gamma}(V)}$$

$$= k\delta'(\theta-\theta')\delta(X-Z) + \epsilon\delta(\theta-\theta')\delta'(X-Z)$$

$$- \sum_{s\geqslant 1} \epsilon^s \lambda^{\mu L}_{(s)}(\theta, [\mathbf{U}], X) \frac{d^s}{dX^s} \frac{\delta\psi(\theta', [\mathbf{U}], Z)}{\delta U^\mu(X)} - \int_{-\infty}^{+\infty} dW \frac{\delta\psi(\theta, [\mathbf{U}], X)}{\delta U^\mu(W)} \chi^\mu_R(W, Z, \theta')$$

$$L^{II}_{\text{eff}}(\theta, \theta', X, Z) = -\frac{1}{2\epsilon} \int_{-\infty}^{+\infty} dW \psi_\theta(\theta, [\mathbf{U}], X)\, \text{sgn}(X-W) \zeta_R(W, Z, \theta')$$

$$- \frac{1}{2} \int_{-\infty}^{+\infty} dW \psi_\theta(\theta, [\mathbf{U}], X)\, \text{sgn}(X-W)$$

$$\times \left(\omega^\mu_0(W) + \sum_{l\geqslant 1} \epsilon^{2l} \tilde{\omega}^\mu_{(2l)}(W) \right) \frac{d}{dW} \frac{\delta\psi(\theta', [\mathbf{U}], Z)}{\delta U^\mu(W)}$$

$$+ \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} dW \psi_\theta(\theta, [\mathbf{U}], X)\, \text{sgn}(X-W) \sum_{s\geqslant 1} \epsilon^s \kappa^{\mu L}_{(s)}(W) \frac{d^s}{dW^s} \frac{\delta\psi(\theta', [\mathbf{U}], Z)}{\delta U^\mu(W)}$$

$$\alpha^{\nu I}(\theta, X, Y) = \sum_{s\geqslant 1} \epsilon^s \lambda^{\nu L}_{(s)}(\theta, [\mathbf{U}], X) \delta^{(s)}(X-Y)$$

$$\alpha^{\nu II}(\theta, X, Y) = \frac{1}{2} \int_{-\infty}^{+\infty} dW \psi_\theta(\theta, [\mathbf{U}], X) \text{sgn}(X-W) \left(\omega^\nu_0(W) + \sum_{l\geqslant 1} \epsilon^{2l} \tilde{\omega}^\nu_{(2l)}(W) \right) \delta'(W-Y)$$

$$- \frac{1}{2\epsilon} \int_{-\infty}^{+\infty} dW \psi_\theta(\theta, [\mathbf{U}], X)\, \text{sgn}(X-W) \sum_{s\geqslant 1} \epsilon^s \kappa^{\nu L}_{(s)}(W) \delta^{(s)}(W-Y).$$

We now have to prove that the solutions $\beta^\nu(\theta, Z, Y)$ satisfy system (4.36) and have the form represented by (4.32)–(4.33). So let us discuss first the resolvability of system (4.38). According to relations (4.34)–(4.35) we can write the main part (in ϵ) of the operator $\hat{L}^I_{\text{eff}}$ in the form

$$L^I_{\text{eff}(0)}(\theta, \theta', X, Z) = k\delta'(\theta-\theta')\delta(X-Z) + \Phi_{U^\nu}(\theta, \mathbf{U}(X)) \omega^\nu_0(X) \Phi_{\theta'}(\theta', \mathbf{U}(X)) \delta(X-Z).$$

The operator $\hat{L}^I_{\text{eff}(0)}$ gives a set of independent operators at different X where the operator

$$k\delta'(\theta-\theta') + \Phi_{U^\nu}(\theta, \mathbf{U}(X)) \omega^\nu_0(X) \Phi_{\theta'}(\theta', \mathbf{U}(X))$$

has at every X exactly two linearly independent left eigenvectors on the space of periodic functions in θ,

$$\eta_1(\theta, X) = 1, \qquad \eta_2(\theta, X) = \Phi(\theta, \mathbf{U}(X))$$

corresponding to the zero eigenvalues.

The vectors $\eta_1(\theta, X)\delta(V-X)$ and $\eta_2(\theta, X)\delta(V-X)$ give the main parts of the vectors

$$\left.\frac{\delta U^0(V)}{\delta\varphi(\theta, X)}\right|_{\mathcal{K}} = \delta(V-X), \qquad \left.\frac{\delta U^1(V)}{\delta\varphi(\theta, X)}\right|_{\mathcal{K}} = \psi(\theta, [\mathbf{U}], X)\delta(V-X) \tag{4.39}$$

which are the left eigenvectors of the operator $\hat{L}^I_{\text{eff}}$ corresponding to the zero eigenvalues.

It is not difficult to see now that the orthogonality of the values $\{g(\theta, X), U^\nu(Y)\}$ to vectors (4.39) on $\mathcal{K}$ implies the orthogonality of $\alpha^{\nu I}(\theta, X, Y)$ to the same vectors. So we know that (4.38) is a compatible system which can be resolved recursively in all the orders of ϵ. The right-hand part of system (4.38) has the form analogous to (4.32)–(4.33) so it is not difficult to see that all the $\beta^\nu(\theta, Z, Y)$ have the necessary form in this case. Using also the fact $\alpha^{\nu I}(\theta, X, Y) = \mathcal{O}(\epsilon)$ we find that the solutions $\beta^\nu(\theta, Z, Y)$ have exactly the required form

(4.32)–(4.33) being written as formal series in ϵ. Besides that, condition (4.37) can also be derived from a not complicated analysis of system (4.38) by use of the same left eigenvectors of $\hat{L}^{I}_{\text{eff}}$ corresponding to the zero eigenvalues.

Finally, let us prove the relation

$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi} L_{\text{eff}}(\theta, \theta', X, Z)\beta^{\nu}(\theta', Z, Y)\frac{\mathrm{d}\theta'}{2\pi}\,\mathrm{d}Z = \alpha^{\nu}(\theta, X, Y) \tag{4.40}$$

for the $\beta^{\nu}(\theta, Z, Y)$ found from (4.38).

Let us note that the difference in the images of the operators $\hat{L}_{\text{eff}}$ and $\hat{L}^{I}_{\text{eff}}$ for our $\beta^{\nu}(\theta, Z, Y)$ is proportional to $\psi_{\theta}(\theta, [\mathbf{U}], X)$ at every (X, Y). The same is also valid for $\alpha^{\nu II}(\theta, X, Y)$ which is the difference between $\alpha^{\nu}(\theta, X, Y)$ and $\alpha^{\nu I}(\theta, X, Y)$. It is not difficult to check then that relation (4.40) follows from (4.38) and the orthogonality of the values

$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi} L^{II}_{\text{eff}}(\theta, \theta', X, Z)\beta^{\nu}(\theta', Z, Y)\frac{\mathrm{d}\theta'}{2\pi}\mathrm{d}Z - \alpha^{\nu II}(\theta, X, Y)$$

to the vectors $\delta U^{2}(V)/\delta\varphi(\theta, X)$ which takes place for our $\beta^{\nu}(\theta, Z, Y)$.

Using formula (4.31) we can claim now that the restricted Poisson bracket $\{U^{\nu}(X), U^{\mu}(Y)\}_{D}$ has exactly the form (4.13).

Let us recall now that the functionals $I^{\nu} = \int_{-\infty}^{+\infty} U^{\nu}(X)\,\mathrm{d}X$ leave invariant the submanifold $\mathcal{K}$ as was proved in theorem 4.1. This means in particular that the flows generated by I^{ν} on $\mathcal{K}$ coincide with their flows generated in the Dirac Poisson structure on this submanifold. Thus, we find that the functionals I^{ν} play the role of the Hamiltonian functions for the higher deformed Whitham systems (4.5) and, in particular, the functional I^{2} plays the role of the Hamiltonian function for the deformed Whitham system (3.16) after the restriction on $\mathcal{K}$. In the same way the functionals I^{0} and I^{1} play the role of the annihilator and the momentum functional for the restricted Gardner–Zakharov–Faddeev bracket, respectively.

At last, let us say that the proof of the theorem for the case of the Magri bracket repeats completely the proof for the Gardner–Zakharov–Faddeev case. □

Remark 4.1. It is not difficult to see that the main ($\sim\epsilon$) term of the Dirac bracket $\{U^{\nu}(X), U^{\mu}(Y)\}_{D}$ on $\mathcal{K}$ coincides with the Dubrovin–Novikov bracket for the Whitham system given by the 'averaging procedure' described above. The Dubrovin–Novikov bracket obtained from the Gardner–Zakharov–Faddeev bracket and the Magri bracket respectively, are compatible with each other and give a bi-Hamiltonian structure for the pure Whitham system for KdV. However, we cannot claim here the same property for the case of the Dirac brackets obtained as the restrictions of the Gardner–Zakharov–Faddeev bracket and the Magri bracket on $\mathcal{K}$, since the Dirac procedure does not preserve the compatibility of the brackets in the general case.

Remark 4.2. It is not difficult to see that the functional

$$\int_{-\infty}^{+\infty} k(X)\,\mathrm{d}X$$

plays the role of annihilator for the restricted Poisson brackets in the case of both the Gardner–Zakharov–Faddeev bracket and the Magri bracket. This circumstance is connected with the conservation of the value $S(+\infty) - S(-\infty)$ by the flows generated by the 'modified' functionals

$$\int_{-\infty}^{+\infty} \tilde{U}^{\nu}(X)q(X)\,\mathrm{d}X$$

with $q(X)$ having compact support and has a general character for the restricted field-theoretical Poisson brackets.

5. Some remarks on the averaging of the Lagrangian structures

Finally, let us discuss also the averaging of Lagrangian functions for the deformed Whitham systems. We will restrict ourselves here only to the situation of the local Lagrangian functions which was considered first by Whitham [69–71] in connection with the pure Whitham approach.

As is well known, the Gardner–Zakharov–Faddeev bracket corresponds to the local Lagrangian formalism of the KdV equation (3.18) having the form

$$\frac{\delta}{\delta v(X,T)} \iint \left[-v_X v_T - \frac{\epsilon}{3} v_X^3 + \epsilon^2 v_{XX}^2 \right] \mathrm{d}X\, \mathrm{d}T = 0 \tag{5.1}$$

(where $\varphi = \epsilon v_X$), which gives

$$v_{XT} + \epsilon v_X v_{XX} + \epsilon^2 v_{XXXX} = 0. \tag{5.2}$$

We also introduce the Whitham pseudo-phase $\Sigma(X,T)$ and look for the solution of (5.2) having the form

$$v(\theta, X, T) = V^{(\mathrm{tot})}\left(\frac{S(X,T)}{\epsilon} + \theta, X, T \right) + \frac{\Sigma(X,T)}{\epsilon}. \tag{5.3}$$

We require now that $V^{(\mathrm{tot})}(\theta, X, T)$ is a periodic function in θ having the form

$$V^{(\mathrm{tot})}(\theta, X, T) = \sum_{k \geqslant 0} V_{(k)}(\theta, X, T), \tag{5.4}$$

where all the functions $V_{(k)}(\theta, X, T)$ are local functionals of

$$(k = S_X, S_T, n = \Sigma_X, k_X, S_{TX}, n_X, k_{XX}, S_{TXX}, n_{XX}, \ldots)$$

having degree k according to the gradation rule:

(1) all the functions $f(k, S_T, n)$ have degree 0;
(2) the derivatives k_{kX}, S_{TkX}, n_{kX} have degree k;
(3) the degree of the product of functions having certain degrees is equal to the sum of their degrees.

According to the normalization of $\Sigma(X,T)$, we put the conditions

$$\int_0^{2\pi} V_{(k)}(\theta, X, T) \frac{\mathrm{d}\theta}{2\pi} \equiv 0 \tag{5.5}$$

for all $V_{(k)}(\theta, X, T)$.

Let us note that the choice of the parameters (k, S_T, n) instead of (k, A, n) is more convenient for the consideration of the Lagrangian structures in our approach. We also recall that the expression for S_T is given by relation (3.15).

It is easy to see that the form (5.3) gives the form of $\phi(\theta, X, T)$ that we consider and all the functions $V_{(k)}(\theta, X, T)$ are uniquely defined by the terms of series (3.17). Indeed, let us first re-expand series (3.17) according to the new gradation rule, i.e.

$$\phi(\theta, X, T) = \sum_{l \geqslant 0} \epsilon^l \Phi'_{(l)} \left(\frac{S(X,T)}{\epsilon} + \theta, X, T \right),$$

where all $\Phi'_{(l)}$ have degree l according to the rules formulated above.

Then we have

$$k V_{(l)\theta}(\theta, X, T) + V_{(l-1)X}(\theta, X, T) = \Phi'_{(l)}(\theta, X, T), \qquad l \geqslant 1$$

which defines uniquely all $V_{(l)}(\theta, X, T)$ view normalization rule (5.5) and we have

$$\Sigma_X(X,T) = \int_0^{2\pi} \phi(\theta, X, T)\frac{d\theta}{2\pi}.$$

Finally, we can substitute series (5.3) in the Lagrangian principle

$$\delta \int\int\int_0^{2\pi} \mathcal{L}(\theta, X, T)\frac{d\theta}{2\pi} dX\, dT$$

with the Lagrangian density

$$\begin{aligned}
\mathcal{L} = {} & -S_X S_T \big(V_\theta^{(\text{tot})}\big)^2 - \Sigma_X \Sigma_T - \tfrac{1}{3} S_X^3 \big(V_\theta^{(\text{tot})}\big)^3 - \Sigma_X S_X^2 \big(V_\theta^{(\text{tot})}\big)^2 - \tfrac{1}{3}\Sigma_X^3 + S_X^4\big(V_{\theta\theta}^{(\text{tot})}\big)^2 \\
& + \epsilon\big(-S_X V_\theta^{(\text{tot})} V_T^{(\text{tot})} - S_T V_\theta^{(\text{tot})} V_X^{(\text{tot})} - \Sigma_T V_X^{(\text{tot})} - \Sigma_X V_T^{(\text{tot})} \\
& - S_X^2\big(V_\theta^{(\text{tot})}\big)^2 V_X^{(\text{tot})} - 2\Sigma_X S_X V_\theta^{(\text{tot})} V_X^{(\text{tot})} - \Sigma_X^2 V_X^{(\text{tot})} + S_X^3 V_{\theta\theta}^{(\text{tot})} V_{\theta X}^{(\text{tot})}\big) \\
& + \epsilon^2\big(-V_T^{(\text{tot})} V_X^{(\text{tot})} - S_X V_\theta^{(\text{tot})}\big(V_X^{(\text{tot})}\big)^2 - \Sigma_X\big(V_X^{(\text{tot})}\big)^2 + 4S_X^2\big(V_{\theta X}^{(\text{tot})}\big)^2 \\
& + S_{XX}^2\big(V_\theta^{(\text{tot})}\big)^2 + S_X^2 V_{\theta\theta}^{(\text{tot})} V_{XX}^{(\text{tot})} + 4S_X S_{XX} V_\theta^{(\text{tot})} V_{\theta X}^{(\text{tot})} + \Sigma_{XX}^2\big) \\
& + \epsilon^3\big(-\tfrac{1}{3}\big(V_X^{(\text{tot})}\big)^3 + 4S_X V_{\theta X}^{(\text{tot})} V_{XX}^{(\text{tot})} + 2S_{XX} V_\theta^{(\text{tot})} V_{XX}^{(\text{tot})} + 2\Sigma_{XX} V_{XX}^{(\text{tot})}\big) + \epsilon^4\big(V_{XX}^{(\text{tot})}\big)^2.
\end{aligned}$$

The averaged Lagrangian function

$$\langle\mathcal{L}\rangle(X,T) \equiv \int_0^{2\pi} \mathcal{L}(\theta, X, T)\frac{d\theta}{2\pi}$$

can also be represented in the graded form with respect to the parameters ($k = S_X, S_T, n = \Sigma_X$) and the Lagrangian equations

$$\frac{\delta}{\delta S(X,T)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle\mathcal{L}\rangle(X,T)\, dX\, dT, \qquad \frac{\delta}{\delta \Sigma(X,T)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle\mathcal{L}\rangle(X,T)\, dX\, dT \tag{5.6}$$

give a system equivalent to (3.16).

The Hamiltonian formalism in the parameters (k, S_T, n) can be written using the Lagrangian formalism (5.6). We get then the Poisson bracket in the canonical form

$$\{n(X), n(Y)\} = \delta'(X-Y), \qquad \{k(X), J(Y)\} = \delta'(X-Y), \tag{5.7}$$

where $J(X)$ is given by the graded expression

$$J(X) = \frac{\partial\langle\mathcal{L}\rangle}{\partial S_T} - \frac{\partial}{\partial X}\frac{\partial\langle\mathcal{L}\rangle}{\partial S_{TX}} + \frac{\partial^2}{\partial X^2}\frac{\partial\langle\mathcal{L}\rangle}{\partial S_{TXX}} + \cdots = \sum_{s\geqslant 0}(-1)^s \frac{\partial^s}{\partial X^s}\frac{\partial\langle\mathcal{L}\rangle}{\partial S_{TsX}}.$$

The Hamiltonian function is also given by the standard expression

$$H[k, J, n] = \int_{-\infty}^{+\infty} \left(-n(X)\Sigma_T(X) + J(X)S_T(X) - \langle\mathcal{L}\rangle\right) dX.$$

The Hamiltonian structure (5.7) is in fact given by the restriction of the symplectic structure corresponding to the Gardner–Zakharov–Faddeev bracket to the submanifold $\mathcal{K}$ and so gives the canonical form of the restriction of this bracket considered in theorem 4.2. The functional

$$I = \int_{-\infty}^{+\infty} J(X)\, dX$$

gives the third annihilator of the restricted Gardner–Zakharov–Faddeev bracket, so we have here the complete set of the canonical variables. Finally, let us say that the functionals $J(X)$,

H should be re-expanded in the graded form corresponding to the variables (k, A, n) using relation (3.15) to come back to our initial gradation rules[9].

Let us conclude that we believe that the averaging of the symplectic structure is also possible in the case of the Magri bracket. However, the symplectic form is much more complicated in this case and no procedure of such kind of symplectic forms is yet known. Let us mention also that the procedure of the restriction of Poisson and symplectic structures must also be generalized to the so-called weakly non-local structures; however, we will not discuss these questions here.

Acknowledgments

The author is grateful to Professor B A Dubrovin for many fruitful discussions. The work was supported by the grant of President of Russian Federation (MD-4903.2008.2) and RFBR (09-01-92442-KE-a, 09-01-12148-ofi-m).

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